Controller Reduction Using Structurally Balanced Truncation Method with New Closed-Loop Structures

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Abstract—In this paper, we discuss controller reduction using structurally balanced truncation (SBT) method with new closed-loop structures. The most consequential disadvantage in SBT method is the problem for feasibility of Lyapunov inequalities for closed-loop systems and it still remain open to this date. The new closed-loop structures are introduced in order to relax the feasibility. Finally, a numerical example is used to verify the availability of the proposed methods.

I. INTRODUCTION

From the perspectives of system maintenance, implementation cost etc., it is a consequential problem to reduce the high order of a controller which is designed for a high-order real plant. Generally, order reduction of controllers is distinguished from that of models (plants, filters, etc.). In many model reduction problems, the approximation quality of obtained reduced-order models is evaluated with the input-output behavior of themselves. However, in controller reduction problems, that of obtained reduced-order controllers cannot be evaluated with the input-output behavior of the reduced-order controllers themselves. Since a controller is designed and implemented with the goals of stabilizing a plant and/or achieving desired control performance, the reduced-order controller which can stabilize the plant and/or achieve the acceptable performance are required.

In order to consider the closed-loop stability, Enns [1], [2] proposed a new reduction method extended balanced truncation (BT) method [3], [2]. By using this method, controller reduction problems can be reduced to stability-weighted model reduction problems. Subsequently, the weighted reduction method was improved by Lin and Chiu [4], [2] and Wang et al. [5], [2], and weighting functions for controller reduction have been derived from various viewpoints, see [6], [7]. These are the controller reduction methods based on the open-loop structure with the weighting functions. In contrast, structurally balanced truncation (SBT) method proposed by Zhou et al. [8] is the controller reduction method based on the original closed-loop structure. Hence this method can take account of the closed-loop behavior, i.e., both the closed-loop stability and the closed-loop performance ($H_\infty$ performance). By utilizing the structured generalized Gramians which are solutions of Lyapunov inequalities for a closed-loop system and have block diagonal structure, SBT method can obtain the same advantages as BT method: if the reduced-order controller exists, the closed-loop system with the reduced-order controller is stable and an upper error bound can be computed easily. However, this method also has some disadvantages. The most consequential disadvantage is the problem for feasibility of the Lyapunov inequalities. Oh and Kim [9] proved that structured generalized Gramians always exist for weighted model reduction problems using SBT method, however, for the controller reduction problems based on the closed-loop structure, the problem still remain open to this date. In particular, if a plant and a designed controller are strictly proper and unstable transfer functions, then the structured generalized Gramians of the closed-loop system configured with them do not exist.

In this paper, in order to relax feasibility of Lyapunov inequalities for a closed-loop system, we introduce new closed-loop structures for controller reduction using SBT method. The new closed-loop structures are equivalent to the original closed-loop structure and are configured with an extended controller and a contracted controller. Design of these controllers are based on a stabilizing controller which is strictly proper and has lower order than the original controller. Furthermore, a numerical example is used to verify the availability of the proposed methods.

The notation used in this paper is stated as follows: the symbol $I$ denotes the identity matrix with appropriate dimension. The superscript $(\cdot)^T$ denotes the transpose of a real matrix. $\text{diag}(\cdot, \ldots, \cdot)$ denotes a block diagonal matrix or a diagonal matrix. The operator $\text{tr}(\cdot)$ denotes the trace of a square matrix, and $F_u(\cdot), F_l(\cdot)$ and $S(\cdot)$ denote the upper linear fractional transformation, the lower linear fractional transformation and the Redheffer star-product, respectively. The $H_\infty$ norm is denoted by $\| \cdot \|_\infty$ and the set of all rational proper and stable transfer functions is denoted by $R\mathcal{H}_\infty$.

II. STRUCTURALLY BALANCED TRUNCATION METHOD

In this section, SBT method proposed by Zhou et al. [8] is reviewed.
Now consider the closed-loop system shown in Fig. 1 with external inputs \( w \), control inputs \( u \), regulated output \( z \) and measured output \( y \). In this system, the \( n_2 \)th-order generalized plant \( G \) is given by

\[
G = \begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix}
\]

and the \( n_k \)th-order stabilizing controller \( K \) is given by

\[
K = \begin{bmatrix}
A_k & B_k \\
C_k & D_k
\end{bmatrix}.
\]

Then the closed-loop transfer function \( T_{zw} \) form \( w \) to \( z \) is derived as

\[
T_{zw} = \mathcal{F}_1(G, K) \in \mathcal{RH}_\infty
\]

\[
= G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}
\]

\[
=: \begin{bmatrix} A & B \\ C & D \end{bmatrix},
\]

\[
A = \begin{bmatrix} A_0 + B_0L_0C_2 & B_0L_0C_k \\
B_0C_0C_2 & A_k + B_kFD_0C
\end{bmatrix},
\]

\[
B = \begin{bmatrix} B_0L_0C_2 & D_0C_2 \\
D_0C_0C_2 & B_kFD_0C
\end{bmatrix},
\]

\[
C = \begin{bmatrix} C_0 + D_0L_0C_2 & D_0L_0C_k \\
D_0L_0C_2 & D_kFD_0C
\end{bmatrix},
\]

\[
D = \begin{bmatrix} D_0 + D_0L_0C_2 & D_0L_0C_k \\
D_0L_0C_2 & D_kFD_0C
\end{bmatrix},
\]

\[
L = (I - D_0D_2)^{-1}, F = (I - D_2D_2k)^{-1}.
\]

Suppose that there exist a structured generalized controllability Gramian \( \mathcal{P} \) and observability Gramian \( \mathcal{Q} \) such that

\[
\begin{align*}
\mathcal{P} + \mathcal{P}^T + BBB^T & \preceq 0, \\
\mathcal{P} & = \text{diag}(\mathcal{P}_g, \mathcal{P}_k) \geq 0, \\
\mathcal{A}^TQ + QA + C^TC & \preceq 0, \\
Q & = \text{diag}(Q_g, Q_k) \geq 0,
\end{align*}
\]

where \( \mathcal{P}_g \) and \( \mathcal{P}_k \) (also, \( Q_g \) and \( Q_k \)) have the same size as \( A_g \) and \( A_k \), respectively. Then we may obtain a nonsingular matrix \( T \) such that

\[
T^{-1}\mathcal{P}_k(T^{-1})^T = TQ_k T := \Sigma_k,
\]

\[
\Sigma_k = \text{diag}(\Sigma_{k1}, \Sigma_{k2}),
\]

\[
\Sigma_{k1} = \text{diag}(\sigma_1, \ldots, \sigma_{n_k}),
\]

\[
\Sigma_{k2} = \text{diag}(\sigma_{n_k+1}, \ldots, \sigma_{n_k}),
\]

\[
\sigma_1 \geq \cdots \geq \sigma_{n_k} > \sigma_{n_k+1} \geq \cdots \geq \sigma_{n_k}.
\]

Transforming the state space coordinate of \( K \) with \( T \), we obtain the structurally balanced realization of \( K \) partitioned compatibly with \( \Sigma_k = \text{diag}(\Sigma_{k1}, \Sigma_{k2}) \) as

\[
K = \begin{bmatrix}
T^{-1}A_k T & T^{-1}B_k \\
C_k T & D_k
\end{bmatrix}
\]

\[
=: \begin{bmatrix}
A_{k11} & A_{k12} & \tilde{B}_k \\
A_{k21} & A_{k22} & \tilde{B}_k \\
C_k & C_k & D_k
\end{bmatrix}.
\]

The reduced-order controller \( \hat{K} \) with \( \hat{n}_k \)th-order is given by

\[
\hat{K} = \begin{bmatrix}
\hat{A}_k & \hat{B}_k \\
\hat{C}_k & \hat{D}_k
\end{bmatrix}
\]

and the closed-loop system \( \hat{T}_{zw} \) with \( \hat{K} \) is stable, i.e.,

\[
\left|\hat{T}_{zw} - T_{zw}\right| \leq 2\text{tr}(\Sigma_{k2}).
\]

For the additional error norm between \( T_{zw} \) and \( \hat{T}_{zw} \), the following upper bound is available:

\[
(A_g + B_gL_0C_2)\mathcal{P}_g + \mathcal{P}_g(A_g + B_gL_0C_2)^T + (B_g + B_0L_0D_0k)B_0^T \leq 0,
\]

\[
(A_g + B_gL_0C_2)^TQ_g + Q_g(A_g + B_gL_0C_2) + (C_g + D_0L_0D_0C_2)^TQ_g + Q_g(C_g + D_0L_0C_2) \leq 0,
\]

\[
(A_k + B_kFD_0C_2)\mathcal{P}_k + \mathcal{P}_k(A_k + B_kFD_0C_2)^T + (B_kFD_0C_2)^T(B_kFD_0C_2) \leq 0.
\]

Remark 1: In SBT method, the reduction problem based on the open-loop structure is formulated as a special
case of the controller reduction problem based on the closed-loop structure. When \( G \) is given by
\[
G = \begin{bmatrix} 0 & W_o \\ W_i & 0 \end{bmatrix}, \quad W_o \in \mathcal{RH}_\infty, \ W_i \in \mathcal{RH}_\infty
\]
and \( K \in \mathcal{RH}_\infty \), we can derive \( T_{zw} \) as
\[
T_{zw} = F_I(G, K) = W_oKw_i \in \mathcal{RH}_\infty.
\]
In this case, there always exist \( P \) and \( Q \) which are solutions of (4) and (5), respectively, see [9].

III. MAIN RESULTS

In this section, we introduce new closed-loop structures for controller reduction using SBT method. As stated in above section, when \( D_{g22} = 0 \) and \( D_k = 0 \), SBT method cannot be applied to the closed-loop system \( T_{zw} \) in (3) if the generalized plant \( G \) and/or the stabilizing controller \( K \) are unstable. So, by using the new closed-loop structures in place of the original structure in (3), we attempt to relax feasibility of Lyapunov inequalities for a closed-loop system on the assumptions that \( D_{g22} = 0 \), \( D_k = 0 \) and the original controller \( K \) is stable. On these assumptions, the closed-loop transfer function \( T_{zw} \) in (3) can be written as
\[
T_{zw} = \begin{bmatrix} A_{g1} & B_{g1}C_k \\ B_{g1}C_{g2} & A_k & B_{g1}D_{g21} \\ C_{g1} & D_{g12}C_k & B_{g1}D_{g21} & D_{g11} \end{bmatrix} \in \mathcal{RH}_\infty. \quad (15)
\]
Now consider a stabilizing controller \( K_d \) which is strictly proper and has lower order than \( K \). Using the stabilizing controller \( K_d \) the state space realization of which is given by
\[
K_d = \begin{bmatrix} A_{kd} & B_{kd} \\ C_{kd} & 0 \end{bmatrix}, \quad (16)
\]
we present the new closed-loop structures below.

A. The closed-loop structure with an extended controller

Suppose that the stabilizing controller \( K_d \) is stable. Define
\[
H = \begin{bmatrix} K_d & I \\ I & 0 \end{bmatrix}, \quad (17)
\]
\[
J = \begin{bmatrix} -K_d & I \\ I & 0 \end{bmatrix}. \quad (18)
\]
Then we can configure the following transfer functions with \( H \) and \( J \):
\[
G_e = S(G, H) = F_I(G, K_d) = G_{12}(I - K_dG_{22})^{-1} = F_I(H, G_{22}) = \begin{bmatrix} A_{g1} & B_{g1}C_k \\ B_{g1}C_{g2} & A_k & B_{g1}D_{g21} \\ C_{g1} & D_{g12}C_k & B_{g1}D_{g21} & D_{g11} \end{bmatrix} = \begin{bmatrix} A_{ge} & B_{ge1} & B_{ge2} \\ C_{ge1} & D_{ge11} & D_{ge21} \\ C_{ge2} & D_{ge12} & D_{ge21} \end{bmatrix}. \quad (19)
\]

Since \( K_d \) is a stabilizing controller for \( G \) and stable, \( G_e \) and \( K_e \) are stable. The closed-loop transfer function \( T_{zw} \) shown in Fig. 2 is given by
\[
T_{zw} = F_I(G_e, K_e) = \begin{bmatrix} A_{ge} & B_{ge1}C_{ke} & B_{ge1}D_{ge21} \\ B_{ke}C_{ge2} & A_{ke} & B_{ke}D_{ge21} \\ C_{ge1} & D_{ge12}C_{ke} & D_{ge11} \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}. \quad (20)
\]

with \( G_e \) and \( K_e \). From Fig. 2, it is easy to see that \( F_I(G, K) \equiv F_I(G_e, K_e) \) and \( F_I(G_e, K_e) \in \mathcal{RH}_\infty \).

Now, in order to improve feasibility of Lyapunov inequalities for a closed-loop system, we consider utilizing the new structure in (21) instead of the original structure in (3).

Defining \( G_e \) as a generalized plant, we can regard \( K_e \) as an extended controller. Using the closed-loop structure in (21), we can select the following Lyapunov inequalities:
\[
\begin{align}
\tilde{A} \tilde{P} + \tilde{P} \tilde{A}^T + \tilde{B} \tilde{B}^T &\leq 0, \\
\tilde{P} &= \text{diag}(\tilde{P}_{ge}, \tilde{P}_{ke}) \geq 0,
\end{align}
\]
where \( \tilde{P}_{ge} \) and \( \tilde{P}_{ke} \) (also, \( \tilde{Q}_{ge} \) and \( \tilde{Q}_{ke} \)) have the same size as \( A_{ge} \) and \( A_{ke} \), respectively. Since \( A_{ge} \) and \( A_{ke} \) are stable matrices, even though \( G \) is unstable, these Lyapunov inequalities can satisfy the necessary conditions corresponding to (11) – (14).

The controller reduction procedure with the closed-loop structure in (21) is now summarized as follows:
Procedure I (based on the structure in (21)): 

1) Find the stabilizing controller $K_d$ for the generalized plant $G$ and configure the closed-loop structure in (21) with the generalized plant $G_e$ in (19) and the extended controller $K_e$ in (20).

2) Find the structured generalized Gramians $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ satisfying the Lyapunov inequalities (22) and (23), if they exist. Similarly to [8] and [9], these are selected such that $\text{tr}(\tilde{\mathcal{P}}_{kc})$ and $\text{tr}(\tilde{\mathcal{Q}}_{kc})$ are minimal.

3) Compute the state transformation matrix $\tilde{T}$ to equalize and diagonalize $\tilde{\mathcal{P}}_{kc}$ and $\tilde{\mathcal{Q}}_{kc}$.

4) Using $\tilde{T}$, we obtain the structurally balanced realization of $K_e$ and the reduced-order controller $\hat{K}_e$ for $K_e$. Eventually, the reduced-order controller $\hat{K}$ for $K$ is given by

$$\hat{K} = K_d + \hat{K}_e.$$  

Remark 2: Several ways can be considered to find the stabilizing controller $K_d$. The simplest way is to choose a reduced-order stabilizing controller obtained with other reduction methods as $K_d$.

Remark 3: Note that the realization of (21) is not minimal and the order of $\hat{K}$ in (24) may be higher than the order of the original controller $K$. Let $n_{kd}$ be the order of $K_d$ and $\hat{n}_{ke}$ be the order of $K_e$. If $n_k \leq n_{kd} + \hat{n}_{ke}$, then $\hat{K}$ in (24) is not a reduced-order controller for $K$. Thus $n_{kd}$ must be lower than $n_k - n_{kd}$.

Remark 4: The new closed-loop structure in (21) was introduced on the assumptions that $D_{g22} = 0$, $D_k = 0$, and $K$ is stable. However, we can except the assumptions that $D_k = 0$ and $K$ is unstable, we can decompose $K$ as

$$K = K_+ + K_-,$$

where $K_+$ is the antistable part of $K$ and $K_-$ is the stable part of $K$. Then the generalized plant $G'$ shown in Fig. 3 is given by

$$G' = \mathcal{S}(G, \left[ \begin{array}{cc} K_+ & I \\ K_- & 0 \end{array} \right]) \in \mathcal{RH}_\infty$$

$$= \left[ \begin{array}{cc} G_{11}' & G_{12}' \\ G_{21}' & G_{22}' \end{array} \right] =: \left[ \begin{array}{c} A_{g'} \\ B_{g'} \\ C_{g'} \\ D_{g'} \end{array} \right],$$

where

$$A_{g'} = \left[ \begin{array}{ccc} A_g + B_gD_kC_{g2} & B_{g2} & B_{g2}C_{k+} \\ B_{k+}C_{g2} & A_{k+} \end{array} \right],$$

$$B_{g'} = \left[ \begin{array}{c} B_{g1} + B_{g2}D_kD_{g21} \\ B_{k+}D_{g21} \end{array} \right],$$

$$C_{g'} = \left[ \begin{array}{c} C_{g1} + D_{g12}D_kC_{g2} \\ C_{g2} \end{array} \right],$$

$$D_{g'} = \left[ \begin{array}{c} D_{g11} + D_{g12}D_kD_{g21} \\ D_{g21} \end{array} \right].$$

It is easy to shown that $\mathcal{F}_I(G, K) \equiv \mathcal{F}_I(G', K_+).$ Since $G'_{22}$ is strictly proper and $K_+$ is strictly proper and stable, we can configure the closed-loop structure as in (21) by using $G'$ and $K_+$ in place of $G$ and $K$, respectively. Then the reduced-order controller $\hat{K}$ is given by

$$\hat{K} = K_+ + K_d + \hat{K}_e,$$

where $\hat{K}_e$ is the reduced-order controller for $K_e = \mathcal{F}_I(J, K_-)$. 

B. The closed-loop structure with a contracted controller

Assume that $T_d$ is a state transformation matrix for the controller $K$ such that

$$K = \left[ \begin{array}{c} T_d^{-1}A_dT_d \\ C_dT_d \end{array} \right] = \left[ \begin{array}{c} A_{kd} \\ C_{kd} \end{array} \right],$$

and $A_{kc}$ is a stable matrix. Define

$$G_e = \left[ \begin{array}{cc} A_{ge} & B_{ge1} \\ C_{ge1} & D_{ge11} \\ C_{ge2} & D_{ge12} \end{array} \right],$$

where $B_{gc2}$ and $C_{gc2}$ are given such that

$$A_{k_{c12}} = B_{gc2}C_{kc},$$

$$A_{k_{c21}} = B_{kc}C_{gc2}.$$ 

Then the transfer function $T_{zw} = \mathcal{F}_I(G_e, K_e)$ is given by

$$T_{zw} = \mathcal{F}_I(G_e, K_e) = \left[ \begin{array}{c} A_{gc} \\ B_{gc1} \\ C_{gc1} \end{array} \right] = \left[ \begin{array}{c} A_g \\ B_{gc2}C_{kc} \\ B_{gc1} \end{array} \right].$$

$$= \left[ \begin{array}{c} B_{gc1}C_{kc} \\ A_{k_{c12}} \\ B_{k_{c21}} \end{array} \right] = \left[ \begin{array}{c} B_{gc1}C_{gc2} \\ A_{k_{c21}} \\ B_{k_{c21}} \end{array} \right].$$

where $B_{gc2}$ and $C_{gc2}$ are given such that

$$A_{k_{c12}} = B_{gc2}C_{kc},$$

$$A_{k_{c21}} = B_{kc}C_{gc2}.\]
It is easy to see that \( F_l(G_c, K_c) = F_l(G, K) \) and \( G_c \in RH_{\infty} \) since \( K_d \) is a stabilizing controller for \( G \).

Now defining \( G_c \) as a generalized plant, we can regard \( K_c \) as a contracted controller. Since \( A_{gc} \) and \( A_{kc} \) are stable matrices, we also select the following Lyapunov inequalities:

\[
\begin{align*}
\dot{P} + \dot{P}^T + BB^T &\leq 0, \\
\dot{Q} + \dot{Q}^T + CC^T &\leq 0,
\end{align*}
\]

where \( \dot{P}_{ge} \) and \( \dot{P}_{ke} \) (also, \( \dot{Q}_{ge} \) and \( \dot{Q}_{ke} \)) have the same size as \( A_{gc} \) and \( A_{kc} \), respectively.

The controller reduction procedure with the closed-loop structure in (33) is now summarized as follows:

**Procedure 2 (based on the structure in (33)):**

1. Find the state transformation matrix \( T_d \) to obtain the realization of the controller \( K \) in (28) and configure the closed-loop structure in (33) with the generalized plant \( G_c \) in (29) and the contracted controller \( K_c \) in (30).
2. Find the structured generalized Gramians \( \mathcal{P} \) and \( \mathcal{Q} \) satisfying the Lyapunov inequalities (34) and (35), if they exist. Similarly to [8] and [9], these are selected such that \( tr(\mathcal{P}_{ke}) \) and \( tr(\mathcal{Q}_{ke}) \) are minimal.
3. Compute the state transformation matrix \( T \) to equalize and diagonalize \( \mathcal{P}_{ke} \) and \( \mathcal{Q}_{ke} \).
4. Using \( T \), we obtain the structurally balanced realization of \( K_c \) and the reduced-order controller

\[
\hat{K}_c = \begin{bmatrix} \hat{A}_{kc} & \hat{B}_{kc} \\ \hat{C}_{kc} & 0 \end{bmatrix}
\]

for \( K_c \). Eventually, the reduced-order controller \( \hat{K} \) for \( K \) is given by

\[
\hat{K} = \begin{bmatrix} A_{kd} & B_{y22}C_{kc} & B_{kd} \\ B_{ke}C_{y22} & A_{kc} & B_{ke} \\ C_{kd} & C_{ke} & 0 \end{bmatrix}.
\]

**Remark 5:** Several ways can be considered to find the state transformation matrix \( T_d \). When the controller \( K \) is stable, the simplest way is to choose a state transformation matrix obtained with a weighted reduction method (such as Enns’s method, Oh and Kim’s method, etc.) as \( T_d \). For instance, consider Enns’s method with an output-sided stability-weighting function (see [1], [6], [7]). Let \( T_s \) be the state transformation matrix obtained with Enns’s method. Transforming the state space coordinate of \( K \) with \( T_s \), we obtain the weighted balanced realization

\[
K = \begin{bmatrix} T_{s}^{-1}A_{k}T_{s} & T_{s}^{-1}B_{k} \\ C_{k}T_{s} & 0 \end{bmatrix} = \begin{bmatrix} A_{k1} & A_{k12} & B_{k1} \\ A_{k21} & A_{k22} & B_{k2} \\ C_{k1} & C_{k2} & 0 \end{bmatrix}.
\]

**IV. NUMERICAL EXAMPLES**

In this section, a numerical example is used to validate the effectiveness of the proposed methods (Procedure 1 and Procedure 2).

We consider the generalized plant \( G \) shown in Fig. 4. The plant \( P \) and the weighting functions \( W_{z1} \) and \( W_{z2} \) in Fig. 4 are given as follows:

\[
P = \begin{bmatrix} 1.370 & 0.000 & 0.000 & 1.000 \\ 2.640 & -3.700 & -1.691 & 1.000 \\ 0.000 & 1.691 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.591 & 0.000 \end{bmatrix},
\]

\[
W_{z1} = \begin{bmatrix} -2.430 & -0.900 \end{bmatrix},
\]

\[
W_{z2} = \begin{bmatrix} -4.620 & -1.897 \\ 1.897 & 1.000 \end{bmatrix}.
\]

In this example, we design a central \( H_{\infty} \) controller \( K \) for \( G \). The central \( H_{\infty} \) controller which satisfies

\[
||T_{zw}||_{\infty} = ||F_l(G, K)||_{\infty} < 4.624
\]

is obtained as follows:

![Diagram](image-url)
TABLE I

<table>
<thead>
<tr>
<th>Order of $K$</th>
<th>Bound $|T_{zw} - T_{zw}^{\hat{}}|_{\infty}$</th>
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<th>Bound $|T_{zw} - T_{zw}^{\hat{}}|_{\infty}$</th>
<th>Bound $|T_{zw} - T_{zw}^{\hat{}}|_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.948 8.280 77.14</td>
<td>3.948 8.282 66.73</td>
<td>3.948 8.282</td>
<td>8.284</td>
</tr>
<tr>
<td>2</td>
<td>8.587 10.47 37.90</td>
<td>0.941 5.545 29.69</td>
<td>0.837 5.430</td>
<td>4.663</td>
</tr>
<tr>
<td>3</td>
<td>0.117 4.716 9.273</td>
<td>0.067 4.675 5.223</td>
<td>0.047 4.663</td>
<td>4.663</td>
</tr>
<tr>
<td>4</td>
<td>0.001 4.626 3.052</td>
<td>0.056 4.641 0.001</td>
<td>0.001 4.626</td>
<td>4.626</td>
</tr>
</tbody>
</table>

Bound: The upper bound of the error norm $\|T_{zw} - T_{zw}^{\hat{}}\|_{\infty}$.

$K = \begin{bmatrix} A_k & B_k \\ C_k & 0 \end{bmatrix}$,

$A_k = [-2.430 0.000 0.000 0.000 0.000, -21.77 -81.71 -103.2 -10.77 -26.05, -11.47 -40.63 -53.03 -5.675 -19.75, -11.47 -40.63 -51.76 -9.375 -17.64]$, $B_k = [-0.900 0.000 10.18 3.754 4.634]^{T}$, $C_k = [11.47 40.63 54.40 5.675 13.73]$.

$H_{\infty}$ norm of the closed-loop system $T_{zw}$ is 4.623. Thus $K$ satisfies the design specification $\|T_{zw}^{\hat{}}\|_{\infty} < 4.624$.

In order to compare, the order of $K$ is reduced with the following controller reduction methods:

- Procedure 1
- Procedure 2

where $W_s$ is given by

$$W_s = (I + PK)^{-1}P.$$  

(41)

By using FWSBT method, we can obtain the stabilizing controller $K_d$ and the state transformation matrix $T_d$ as follows:

$$K_d = \begin{bmatrix} -136.3 & -29.55 \\ -21.13 & 0.000 \end{bmatrix},$$

$$T_d = \begin{bmatrix} 0.059 & -0.966 & 0.275 & 0.000 & 6.445 \\ 0.310 & -2.131 & 3.582 & -1.408 & 0.057 \\ -0.542 & 1.364 & -2.281 & 0.892 & -1.197 \\ -0.103 & -0.728 & 0.205 & -4.987 & -0.104 \\ -0.161 & -1.561 & -1.754 & 0.829 & -0.768 \end{bmatrix}.$$  

Since the plant $P$ has an unstable pole at 1.370, the structured generalized Gramians satisfying the Lyapunov inequalities (4) and (5) do not exist, i.e., we cannot apply SBT method to the original closed-loop structure in (3). However, by using Procedure 1 and Procedure 2, we can obtain the reduced-order controllers. The results are listed in Table I.

Table I shows that the proposed methods, Procedure 1 and Procedure 2, guarantee the closed-loop stability and the closed-loop performance ($H_{\infty}$ performance) by the error bounds and work very well in the reduced-order controllers with 2nd-order and 3rd-order. Hence the proposed methods can use the advantages of SBT method in the example which generally it cannot be applied to. Since FWSBT method is the controller reduction method based on the open-loop structure with the stability-weighting function, it can take account of the closed-loop stability, however, error bounds do not obtained, i.e. the closed-loop performance degradation cannot be estimated.

The results also show that the error bounds obtained with the proposed methods are conservative. There is the problem for conservativeness of the error bound as another disadvantage of SBT method. The conservativeness relates strongly to the selection method for structured generalized Gramians (the step 2) in Procedure 1 and Procedure 2. Hence we will have to improve the selection method to obtain more tight error bounds.

V. CONCLUSION

In this paper, in order to relax feasibility of Lyapunov inequalities in SBT method, we have introduced the new closed-loop structures in (21) and (33). The Lyapunov inequalities (22) and (23) based on the new structure in (21) (also, the Lyapunov inequalities (34) and (35) based on the new structure in (33)) can satisfy the necessary conditions corresponding to (11) – (14). A numerical example have been used to verify the availability of the proposed methods (Procedure 1 and Procedure 2). The results have shown that the proposed methods can utilize the advantages of SBT method in the example which generally it cannot be applied to.

REFERENCES
