Uncertain Frequency Domain Estimates for LTV Discrete-Time Systems

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Abstract— Although upper bound estimation or the worst case gain analysis are well-known methods for linear time invariant (LTI) systems, they cannot be applied directly for linear time varying systems (LTV). The main aim of the paper is to extend the methodology to the LTV case. Our approach base on approximated magnitude-frequency response using SVD-DFT method for LTV systems. The upper bound is estimated using geometrical approach for complex numbers of the approximated magnitude-frequency response. Our considerations begins from description of discrete-time LTV state space model with additive perturbation of the system operator. Further the relationship for system additive perturbation operator is proved as well as it is shown how the results can be employed for estimation of the upper bound on magnitude diagram. Theoretical considerations are complemented by numerical example – upper bound analysis for uncertain variable structure system.

I. INTRODUCTION

The frequency response magnitude of an uncertain system generally depends on the values of its uncertain elements. A point wise-in-frequency worst-case gain analysis yields the frequency-dependent curve of maximum magnitude or equivalently the upper bound on magnitude-frequency response. Determining the gain over all allowable values of the uncertain elements is referred to as a worst-case gain analysis. This maximum gain is called the worst-case gain.

Frequency bounds are an essential step in frequency domain robust control design methods. It is involved in the whole design procedure including modelling, system analysis, controller design and final design evaluation [1], [2], [3], [4]. Unfortunately well-developed concepts and analytic methods of time-invariant systems are not yet applicable to linear time-variant systems (LTV) and systems defined on finite time horizon.

Main aim of this paper is to propose the method for estimating the point wise-in-frequency worst-case gain for LTV systems defined on finite time horizon. Frequency domain transformation for time-varying system can be done using method reported in earlier papers [5], [6]. It has been shown that proposed Bode diagrams are efficient way for description of time-varying dynamical systems. They have many properties of classical Bode diagrams and are almost the same for LTI systems [5], [6].

II. STATE SPACE AND OPERATORS BASED MODEL DESCRIPTION

Uncertain time varying discrete-time systems, can be described employing state equations with time-dependent matrices and additive perturbation matrices in the following form

\[
\begin{align*}
    x_n(k+1) &= (A(k) + \Delta_n) x_n(k) + (B(k) + \Delta_u) u(k), \quad (1) \\
    y_n(k) &= (C(k) + \Delta_c) x_n(k) + (D(k) + \Delta_d) u(k) \quad (2)
\end{align*}
\]

System (1-2) is called perturbed since at least one of perturbation matrices is non zero matrix. In case of all perturbation matrices are equal to zero, the system is called nominal and can be described by following equations

\[
\begin{align*}
    x_n(k+1) &= A(k) x_n(k) + B(k) u(k), \quad (3) \\
    y_n(k) &= C(k) x_n(k) + D(k) u(k) \quad (4)
\end{align*}
\]

where \( \{x_n(k), x_n(k) \in \mathbb{R}^n\} \) is nominal, perturbed state, \( \{u(k) \in \mathbb{R}^n\} \) is nominal control, \( \{y_n(k), y_n(k) \in \mathbb{R}^p\} \) is nominal, perturbed output, and \( A(k), \Delta_n \in \mathbb{R}^{n \times n} \), \( B(k), \Delta_u \in \mathbb{R}^{n \times p} \), \( C(k), \Delta_c \in \mathbb{R}^{p \times n} \), \( D(k), \Delta_d \in \mathbb{R}^{p \times p} \) are system matrices with their perturbations and \( k \in \{0, \ldots, N-1\} \).

Output and state trajectories are denoted by index \( p \) or \( \Delta \), which allow to distinguish between nominal (\( p \)) and perturbed (\( \Delta \)) trajectory. Model of the system can be described with the help of following operators.

\[
\begin{align*}
    \hat{A} &= \begin{bmatrix}
    0 & 0 & \cdots & 0 & 0 \\
    1 & 0 & \cdots & 0 & 0 \\
    & \ddots & \ddots & \ddots & \ddots \\
    & & \ddots & \ddots & \ddots \\
    N-1 & \cdots & A(N-2) & A(N-1) & 1
    \end{bmatrix} \quad (5) \\
    \hat{B} &= \begin{bmatrix}
    B(0) & 0 & 0 \\
    0 & \ddots & \ddots \\
    0 & 0 & B(N-1)
    \end{bmatrix} \quad (6) \\
    \hat{C} &= \begin{bmatrix}
    C(0) & 0 & 0 \\
    0 & \ddots & \ddots \\
    0 & 0 & C(N-1)
    \end{bmatrix} \quad (7)
\end{align*}
\]
where operators $\hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}$ have block diagonal form.

$$
\begin{align*}
\hat{x}_\lambda &= \begin{bmatrix} x_\lambda(0) \\ \vdots \\ x_\lambda(N-1) \end{bmatrix}, \\
\hat{y}_\lambda &= \begin{bmatrix} y_\lambda(0) \\ \vdots \\ y_\lambda(N-1) \end{bmatrix}, \\
\hat{u} &= \begin{bmatrix} u(0) \\ \vdots \\ u(N-1) \end{bmatrix}
\end{align*}
$$

(9)

$\hat{\mathbf{x}}_\lambda, \hat{\mathbf{y}}_\lambda$, output $\hat{\mathbf{y}}_\lambda$, and input $\hat{\mathbf{u}}$ have block column form.

Then output trajectory of nominal system can be given in the following form

$$
\hat{y} = \hat{\mathbf{C}}\hat{\mathbf{N}}x_\lambda + (\hat{\mathbf{C}}\hat{\mathbf{L}}\hat{\mathbf{B}} + \hat{\mathbf{D}})\hat{u} = \hat{\mathbf{C}}\hat{\mathbf{N}}x_\lambda + \hat{\mathbf{H}}\hat{u}
$$

(11)

In order to estimate worst case gain of input-output operator $\hat{\mathbf{H}}$ it is assumed that initial conditions are equal to zero. The system response at zero initial conditions is determined by the $\hat{y}_\lambda = \hat{\mathbf{H}}\hat{u}$ term. The operator $\hat{\mathbf{H}} = \hat{\mathbf{C}}\hat{\mathbf{L}}\hat{\mathbf{B}} + \hat{\mathbf{D}}$ is a compact, Hilbert-Schmidt operator from $l_2$ into $l_2$ and actually maps boundedly signals $\hat{u} \in \mathcal{U}$ into signals $\hat{y} \in \mathcal{Y}$.

Symbol $\|\hat{\mathbf{L}}\|$ denotes the Euclidean norm of matrix operator $\hat{\mathbf{L}}$ defined as

$$
\|\hat{\mathbf{L}}\| = \sigma_{\text{max}}(\hat{\mathbf{L}}) = \sqrt{\lambda_{\text{max}}(\hat{\mathbf{L}}^\dagger \hat{\mathbf{L}})} = \sup_{\hat{\mathbf{H}}\neq 0} \|\hat{\mathbf{L}}\hat{\mathbf{H}}\| = \sup_{\hat{\mathbf{H}}\neq 0} \sqrt{\langle \hat{\mathbf{H}}^\dagger \hat{\mathbf{L}}\hat{\mathbf{H}} \rangle}
$$

(12)

and vector norm is induced by inner product i.e.

$$
\|\hat{\mathbf{x}}\| = \sqrt{\sum_j \hat{x}_j^2} = \sqrt{\langle \hat{\mathbf{x}}, \hat{\mathbf{x}} \rangle}
$$

(13)

Matrix inequality is understood in following sense:

$$
\hat{\lambda}^2 \hat{\lambda} \leq \hat{\mathbf{M}} \Leftrightarrow \hat{x}^\dagger (\hat{\lambda}^2 \hat{\lambda}) \hat{x} \leq \hat{x}^\dagger \hat{\mathbf{M}} \hat{x}
$$

(14)

Additive perturbations introduced by eq. (1-2) can be transformed into matrix operators in similar manner as for matrices $\mathbf{B}(k), \mathbf{C}(k)$ and $\mathbf{D}(k)$, i.e. using block diagonal form to define new four operators $\hat{\lambda}\lambda, \hat{\lambda}_b, \hat{\lambda}_c, \hat{\lambda}_n$. Each of them satisfy following additivity condition:

$$
\hat{\lambda}\lambda = \hat{\lambda} + \hat{\lambda}\lambda, \quad \hat{\mathbf{B}}_\lambda = \hat{\mathbf{B}} + \hat{\mathbf{B}}_\lambda, \quad \hat{\mathbf{C}}_\lambda = \hat{\mathbf{C}} + \hat{\mathbf{C}}_\lambda, \quad \hat{\mathbf{D}}_\lambda = \hat{\mathbf{D}} + \hat{\mathbf{D}}_\lambda
$$

(15)

We assume that similar notation can be also applied to represent uncertainty in the system operator:

$$
\hat{\mathbf{H}}_\lambda = \hat{\mathbf{H}} + \hat{\lambda}_n
$$

(16)

what we prove in the consecutive section.

III. SYSTEM ADDITIVE PERTURBATION OPERATOR

**Theorem 1.** System additive perturbation operator $\hat{\lambda}_n$ satisfying (16) where system matrices are given by (15) under condition $\|\hat{\lambda}_n\| < 1$ take following form:

$$
\begin{align*}
\hat{\lambda}_n &= \hat{\mathbf{C}}_\lambda \left(\hat{\lambda}_\lambda + \left(\hat{\lambda}_\lambda \right)^2 + \ldots \right)\hat{\mathbf{B}}_\lambda \\
&+ \hat{\mathbf{C}}_\lambda \hat{\lambda}_n + \hat{\mathbf{C}}_\lambda \hat{\mathbf{B}} + \hat{\lambda}_n \\
&+ \hat{\mathbf{C}}_\lambda \left(\hat{\lambda}_\lambda + \left(\hat{\lambda}_\lambda \right)^2 + \ldots \right) \hat{\mathbf{B}}_\lambda + \hat{\mathbf{D}}_\lambda + \hat{\lambda}_n
\end{align*}
$$

(17)

Upper index denotes number of Taylor series terms, in this case infinity.

**Proof.** The nominal state trajectory in case of zero initial conditions can be written as follows

$$
\hat{x}_\lambda = \hat{\mathbf{L}}\hat{\mathbf{B}}\hat{\mathbf{u}}
$$

(18)

It has been proven [7] that the perturbed state trajectory can be written by

$$
\hat{x}_\lambda = \hat{x}_\lambda + \hat{\lambda}_\lambda \hat{x}_\lambda + \hat{\lambda}_n \hat{\mathbf{u}}
$$

(19)

Assuming that $(\hat{\mathbf{I}} - \hat{\lambda}_\lambda \hat{\mathbf{I}})$ is invertible, where $\hat{\mathbf{I}}$ is identity operator (matrix), we can rearrange above equation into following form

$$
\hat{x}_\lambda = (\hat{\mathbf{I}} - \hat{\lambda}_\lambda \hat{\mathbf{I}})^{-1} (\hat{\mathbf{I}}\hat{\mathbf{B}}\hat{\mathbf{u}} + \hat{\lambda}_n \hat{\mathbf{u}})
$$

(20)

and the perturbed output trajectory is given by

$$
\hat{\mathbf{y}}_\lambda = \left(\hat{\mathbf{C}} + \hat{\lambda}_c \hat{\mathbf{C}}\right) \left(\hat{\mathbf{I}} - \hat{\lambda}_\lambda \hat{\mathbf{I}}\right)^{-1} \hat{\mathbf{I}}\hat{\mathbf{B}}\hat{\mathbf{u}} + \hat{\lambda}_n \hat{\mathbf{u}}
$$

(21)

Necessary condition for invertibility of $\left(\hat{\mathbf{I}} - \hat{\lambda}_\lambda \hat{\mathbf{I}}\right)$ is $\|\hat{\lambda}_\lambda\| < 1$. It is also possible to substitute the inverse term by the following expansion of the Taylor series

$$
\left(\hat{\mathbf{I}} - \hat{\lambda}_\lambda \hat{\mathbf{I}}\right)^{-1} = \mathbf{I} + \hat{\lambda}_\lambda \hat{\mathbf{I}} + \left(\hat{\lambda}_\lambda \hat{\mathbf{I}}\right)^2 + \ldots \quad \text{and then}
$$

$$
\hat{\mathbf{y}}_\lambda = \left(\hat{\mathbf{C}} + \hat{\lambda}_c \hat{\mathbf{C}}\right) \left(\mathbf{I} + \hat{\lambda}_\lambda \hat{\mathbf{I}} + \left(\hat{\lambda}_\lambda \hat{\mathbf{I}}\right)^2 + \ldots \right) \hat{\mathbf{I}}\hat{\mathbf{B}}\hat{\mathbf{u}} + \hat{\lambda}_n \hat{\mathbf{u}}
$$

(22)

The output perturbed trajectory can be also written in terms of perturbed operators

$$
\hat{\mathbf{y}}_\lambda = \left(\hat{\mathbf{C}}_\lambda \hat{\lambda}_\lambda \hat{\mathbf{B}}_\lambda + \hat{\lambda}_n \hat{\mathbf{B}}_\lambda\right)\hat{\mathbf{u}} = \hat{\mathbf{H}}_\lambda \hat{\mathbf{u}}
$$

(23)

thus the perturbed operator is given by

$$
\hat{\mathbf{H}}_\lambda = \hat{\mathbf{H}} + \hat{\mathbf{C}}_\lambda \left(\hat{\lambda}_\lambda \right)^2 + \ldots \hat{\mathbf{B}}_\lambda
$$

(24)

and after simplification

$$
\hat{\mathbf{H}}_\lambda = \hat{\mathbf{H}} + \hat{\mathbf{C}}_\lambda \left(\hat{\lambda}_\lambda \right)^2 + \ldots \hat{\mathbf{B}}_\lambda
$$

(25)
Perturbed operator can be written in terms of nominal operator and additive perturbation operator that is equivalent to (17) what finish the proof.

For sufficiently small perturbation matrices we have
\[
\|DL\|^n << 1 \quad \text{for } n \geq 2
\]
and hence all powers higher or equal 2 can be omitted. By restricting the perturbation matrices to the 1st order we obtain the first order approximation of the perturbation operator given in the form
\[
\hat{\Delta}_{(i)}^{(n)} = \hat{C}L\hat{A}_i\hat{L}B + \hat{C}L\hat{A}_n + \hat{\Delta}_c\hat{L}B + \hat{\Delta}_n
\] (26)

Further simplifications are possible under particular assumption of perturbation matrices. We can distinguish four different perturbations bounds.

A. Norm bounded perturbation

The norm bounded uncertainty is the simplest one, that is
\[
\|
\Delta
\| \leq \delta
\] (27)

where \(\delta\) is a known number. For norm bounded perturbation the estimate of perturbation product is given by
\[
\hat{\Delta}^T\hat{\Delta} \leq \delta^2I
\] (28)

where \(I\) the identity matrix. So we can write inequalities for the following norms
\[
\|
\hat{\Delta}_{(i)}^{(n)}
\| \leq \left\|
\hat{C}L\hat{A}_i\hat{L}B + \hat{C}L\hat{A}_n + \hat{\Delta}_c\hat{L}B + \hat{\Delta}_n
\right\| \leq \delta
\] (29)

\[
\left\|
\hat{\Delta}_{(i)}^{(n)}
\right\| \leq \left\|
\hat{C}L\hat{A}_i\right\| + \delta_c + \left\|
\hat{C}L\hat{A}_n + \hat{\Delta}_c\hat{L}B + \hat{\Delta}_n
\right\| + \delta_n
\] (30)

B. Structure bounded perturbation

This type of uncertainty is described as product of following three structure matrices
\[
\hat{\Delta} = D_iX_iD_1
\] (31)

where matrices \(D_i \in \mathbb{R}^{n \times n}\), \(D_1 \in \mathbb{R}^{n \times n}\) are both known, and matrix \(X \in \mathbb{R}^{n \times n}\) is unknown nonnegative definite diagonal square matrix with \(X^2X = I\). In this case we have:
\[
\hat{\Delta}^T\hat{\Delta} = D_i^T X_i^2 D_1^T X_1 D_1 \leq \|D_i\| \|D_1\| X_1 D_1 \leq \|D_i\| \|D_1\| X_1
\] (32)

Let us substitute \(\hat{\Delta}_i = D_iX_iD_1\), \(\hat{\Delta}_n = D_nX_nD_2\), \(\hat{\Delta}_c = D_cX_cD_2\), \(\hat{\Delta}_b = D_bX_bD_2\). It is possible to estimate the norm of perturbation matrix \(\hat{\Delta}_{(i)}^{(n)}\) holding conditions \(\|X_i\| \leq 1\), \(\|X_n\| \leq 1\), \(\|X_c\| \leq 1\), \(\|X_b\| \leq 1\) in following way
\[
\|
\hat{\Delta}_{(i)}^{(n)}
\| \leq \left\|
\hat{C}L\hat{A}_i\right\| \|D_2\| \|L\| \|B\| + \left\|
\hat{C}L\hat{A}_n\right\| \|D_2\| \|L\| \|B\| + \|D_1\| \|D_2\| \|L\| \|B\|
\] (33)

C. Value bounded perturbation

Another type of uncertainty often appears when the elements of the matrix operator \(\hat{\Delta} = \{d_{ij}\}_{i=1}^n\) are unknown but bounded \(|d_{ij}| \leq \overline{d}_{ij}, \ i, j = 1 \ldots n\). Set of upper bounds of matrix \(\hat{\Delta}\) is a new matrix \(D = \{\overline{d}_{ij}\}_{i=1}^n\).

For any \(x \in \mathbb{R}^n\), it can be computed that
\[
x^T\hat{\Delta}^T\hat{\Delta}x = \sum_{i=1}^n \sum_{j=1}^n (\overline{d}_{ij}x_i)^2 \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left(\overline{d}_{ij}^2 + \overline{d}_{ji}^2\right)
\] (34)

leaves
\[
\leq \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \overline{d}_{ij}^2 x_i^2 = \text{trace}(D^T D)\|x\|^2
\]

this implies that
\[
\hat{\Delta}^T\hat{\Delta} \leq \text{trace}(D^T D) I
\] (35)

So it is possible to estimate the norm of perturbation matrix \(\hat{\Delta}_{(i)}^{(n)}\) in following way
\[
\|
\hat{\Delta}_{(i)}^{(n)}
\| \leq \|\hat{C}L\| \text{trace}(D^T D) \|I\| + \|\hat{C}L\| \text{trace}(D^T D) + \|\hat{C}L\| \text{trace}(D^T D)
\]

D. Parameter bounded perturbation of linear combination

Generalization of additive uncertainty given in eq. (26) is uncertainty of linear combination, defined as follows for each perturbation component, e.g. \(\hat{C}L\hat{A}_i\hat{L}B\), \(\hat{C}L\hat{A}_n\hat{L}B\), \(\hat{\Delta}_c\hat{L}B\) and \(\hat{\Delta}_n\).

\[
\hat{\Delta} = d_1D_1 + \ldots + d_nD_n = \sum_{i=1}^n d_i\hat{\Delta}_i
\] (37)

Note that
\[
\hat{\Delta}^T\hat{\Delta} = \sum_{i=1}^n d_i^2D_i^T D_i
\]

\[
\leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left(\overline{d}_{ij}^2 D_j^T D_j + \overline{d}_{ji}^2 D_{ji}^T D_{ji}\right) \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \overline{d}_{ij}^2\]

or alternatively
\[
\hat{\Delta}^T\hat{\Delta} \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \overline{d}_{ij}^2\]

Substituting \(\hat{\Delta}_i = d_i\hat{\Delta}_i\), \(\hat{\Delta}_n = \sum_i d_i\hat{\Delta}_n\), \(\hat{\Delta}_c = \sum_i c\hat{\Delta}_c\)

\(\hat{\Delta}_b = \sum_i d_i\hat{\Delta}_b\) equation (26) can be rewritten in the form
\[
\hat{\Delta}_{(i)}^{(n)} = \sum_{i=1}^n d_i\hat{\Delta}_i + \sum_{i=1}^n \overline{d}_{ij}^2\hat{\Delta}_{(i)}^{(n)}
\] (40)
Applying norm and triangle inequality both sides we have
\[
\|\tilde{\Delta}_u\| \leq \sum_{j=1}^{k} \|\mathcal{C}\tilde{\Lambda}_{\alpha_j} \mathbf{L} \mathbf{B}\| + \sum_{j=1}^{k} \|\mathcal{C}\tilde{\Lambda}_{\nu_j}\| \quad (41)
\]
\[
+ \sum_{i=1}^{l} c_i \|\tilde{\Delta}_c_i \mathbf{L} \mathbf{B}\| + \sum_{i=1}^{l} d_i \|\tilde{\Delta}_b_i\|
\]
or
\[
\|\tilde{\Delta}_u^{(\alpha)}\| \leq \lambda_{\max} \left( \begin{array}{ccc} a_1 \tilde{\Lambda}_{\alpha_1} + a_2 \tilde{\Lambda}_{\alpha_2} \\ + c_1 \tilde{\Delta}_{C_1} + c_2 \tilde{\Delta}_{C_2} \\ \end{array} \right) \left( \begin{array}{c} \mathbf{B} \\ \mathbf{L} + b_1 \tilde{\Delta}_{B_1} \\ + b_2 \tilde{\Delta}_{B_2} \\ \end{array} \right)
\]
\[+ (\tilde{\mathbf{C}} + c_1 \tilde{\Delta}_{C_1} + c_2 \tilde{\Delta}_{C_2}) \mathbf{L} (b_1 \tilde{\Delta}_{B_1} + b_2 \tilde{\Delta}_{B_2})
\]
\[+ (c_1 \tilde{\Delta}_{C_1} + c_2 \tilde{\Delta}_{C_2}) \mathbf{L} \mathbf{B} + d_1 \tilde{\Delta}_{B_1} + d_2 \tilde{\Delta}_{B_2} \]
\[\] (43)

IV. FREQUENCY TRANSFORMATION

Singular values play important role in control systems analysis and design. Foundations of frequency analysis based on SVD-DFT decomposition of the system operator have been explained in [5]. Now it will be recalled only the general result of it.

Bode diagrams include the magnitude-frequency response \( |G(\omega_j)| \), which is uniquely defined by
\[
|G(\omega_j)| = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^j \|\text{DFT}_u[\mathbf{u}_j]\|^2 \quad (44)
\]
and the phase-frequency response \( \varphi(\omega_j) = \arg(G(\omega_j)) \) which can be written as
\[
\varphi(\omega_j) = \arg \left( \sum_{i=1}^{N} \sigma_i^j \|\text{DFT}_u[\mathbf{u}_j]\| \|\text{DFT}_v[\mathbf{v}_j]\| \right) \quad (45)
\]
where \( \text{USV}^T \) is result of Singular Value Decomposition of system input-output operator \( \mathbf{H} \), such that \( S = \text{diag}(\sigma_i) \) is diagonal matrix with singular values and \( \mathbf{U}, \mathbf{V} \) are orthonormal matrices composed of column vectors \( \mathbf{u}_j \) and \( \mathbf{v}_j \) respectively. Singular values \( \sigma_i \) in eqs. (44-45) play their part as weight functions. The derived relationships hold true for both time invariant and time variant systems. Diagrams obtained in the way shown for time invariant systems at a finite time horizon are close to Bode characteristics obtained in the classic way by substituting \( z = \exp(j \cdot \omega \cdot T_p) \) into discrete transfer function.

The magnitude \( |G(j \omega_k)| \) may be interpreted as cumulative amplification of all harmonics in output spectra for a given sinusoidal input.

In general, the frequency diagrams for LTV systems does not satisfy few of the properties that are held for LTI systems, e.g. cascade of two systems cannot correspond to multiplication in the frequency domain of the components as well as the input/output relationship does not take in general the form of a multiplication in the frequency domain. Moreover the method is not additive, but homogeneous only.

For example, following relation is hold for LTI systems.
\[
\mathbf{\hat{H}}_x(\omega) = \mathbf{H}(\omega) + \tilde{\Delta}_u(\omega) \quad (46)
\]
where \( \mathcal{F}(\mathbf{\hat{H}}) = \mathbf{\hat{H}}(\omega) \) transformation defined by equations (44-45). For LTV systems following inequality holds instead:
\[
\|\mathbf{\hat{H}}_x(\omega)\| \leq |\mathbf{\hat{H}}(\omega)| + \tilde{\Delta}_{u_{\text{max}}}(\omega) \quad (47)
\]
and the condition always holds
\[
\lim_{\omega \to \omega_{\text{max}}} [\mathbf{\hat{H}}_x(\omega) - \mathbf{\hat{H}}(\omega) - \tilde{\Delta}_{u_{\text{max}}}(\omega)] = 0 \quad (48)
\]

V. ESTIMATION

The deviation of magnitude and phase diagrams can be estimated using a method based on simple geometrical relationship for complex numbers. It can be summarized with the following theorem.

\textbf{Theorem 2.} Upper bound of the magnitude diagram can be estimated in following way:
\[
\|\mathbf{\hat{H}}_x(\omega)\| \leq |\mathbf{\hat{H}}(\omega)| + \tilde{\Delta}_{u_{\text{max}}}(\omega) \quad (49)
\]
where
\[
v_{u_0}(\omega) = \arg \left( \|\mathcal{C}\tilde{\Lambda}_{\alpha_0} \mathbf{L} \mathbf{B}\|(\omega) \right),
\]
\[
v_{u_B}(\omega) = \arg \left( \|\mathcal{C}\tilde{\Lambda}_{B_0}\|(\omega) \right),
\]
\[
v_{C_0}(\omega) = \arg \left( \|\tilde{\Delta}_C \mathbf{L} \mathbf{B}\|(\omega) \right),
\]
\[
v_{C_B}(\omega) = \arg \left( \|\tilde{\Delta}_B \mathbf{B}\|(\omega) \right)
\]
\[
(\tilde{\Delta}_{u_{\text{max}}})^{(i)}(\omega) = \sum_{i=1}^{k} \frac{1}{\max} \left[ \begin{array}{c} a_i \sgn(\cos(v_{i_0}(\omega) - v(\omega))) \|\mathcal{C}\tilde{\Lambda}_{i_0} \mathbf{L} \mathbf{B}\| \()\omega) \\
+b_i \sgn(\cos(v_{i_B}(\omega) - v(\omega))) \|\tilde{\Delta}_{i_B}\|(\omega) \\
+c_i \sgn(\cos(v_{i_C}(\omega) - v(\omega))) \|\tilde{\Delta}_C \mathbf{L} \mathbf{B}\| \()\omega) \\
+d_i \sgn(\cos(v_{i_B}(\omega) - v(\omega))) \|\tilde{\Delta}_B \mathbf{B}\| \()\omega) \\
\end{array} \right] \quad (51)
\]
\[v(\omega) = \arg \left( \mathbf{H}(\omega) \right) \quad (52)
\]
and, \( \mathbf{\hat{H}}_x(\omega) \neq 0 \) for all \( \omega_k = \frac{2\pi(k-1)}{NT_p} \) and \( k = 1,2,\ldots,\frac{N}{2} \).
Eq. (49) can be easily proven geometrically for any fixed $\omega$, for which DFT can be understood as a complex number. The magnitude of an uncertain system is bounded by parallel perturbation estimate given by eq. (51). The estimate is the worst possible combination of eq. (40) calculated in the direction parallel to operator $\hat{H}(\omega)$. The phase of an uncertain system can be bounded by the orthogonal estimate given by eq. (51).

If $\hat{H}(\omega) = 0$ then the above theorem cannot be used for estimation of the bounds. From a practical point of view such a singular case can be converted to a non-singular case e.g. by introducing additional negligible small parameters perturbation or by changes in sampling period or in the simulation horizon.

**Corollary 1.** For small $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ the perturbation operator $\hat{H}_{\text{max}}$ can be replaced by its first order approximation, thus

$$\max_{\omega} \left( \hat{H}(\omega) \right) \approx \left( \hat{H}(\omega) + \hat{H}_{\text{max}}^{(i)}(\omega) \right)$$

and the operator $\hat{H}$ in eq. (52) can be approximated by the operator $\hat{H}$ and $\nu$ is approximated by $v(\omega) \approx \arg \left( \hat{H}(\omega) \right)$

**VI. NUMERICAL EXAMPLE**

Let us consider discretised variable structure oscillatory-inertial system. Fundamental matrices for system given in the state space form are defined in following way:

(I) oscillatory structure

$$A_f = \begin{bmatrix} -0.21 & 0.66 & -0.31 \\ -0.57 & -0.21 & -0.49 \\ 0.45 & 0.35 & -0.27 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0 \\ -1.19 \\ 0.056 \end{bmatrix}$$

II inertial structure

$$A_i = \begin{bmatrix} 0.071 & 0.25 & 0.14 \\ -0.25 & -0.18 & -0.35 \\ 0.14 & -0.35 & -0.23 \end{bmatrix}, \quad B_i = \begin{bmatrix} -1.17 \\ -0.46 \\ -0.26 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1.47 \\ 0.056 \end{bmatrix}$$

$$D_i = D_f = 0 \text{ for both cases and sampling period } T_p = 0.04 \text{ s.}$$

Real system matrices are linear combination of given above two fundamental matrices and two time dependent coefficients in following way:

$$A(k) = \alpha(k) A_f + \beta(k) A_i$$
$$B(k) = \alpha(k) B_f + \beta(k) B_i$$
$$C(k) = \alpha(k) C_f + \beta(k) C_i$$
$$D(k) = \alpha(k) D_f + \beta(k) D_i$$

$$\alpha(k) = \left( \frac{N-k}{N-1} \right)^k, \quad \beta(k) = \left( \frac{k-1}{N-1} \right)^k$$

where $k = 1, 2, \ldots, N$, $N = 100$.

Perturbation matrices $\Delta_A, \Delta_B, \Delta_C, \Delta_D$ are given as following parameter bounded perturbation of linear combination:

$$\Delta_A = a_1 \begin{bmatrix} -7 & 8 & 0 \\ -3 & 2 & 7 \end{bmatrix} + a_2 \begin{bmatrix} 2 & 2 & 3 \\ 0 & 4 & 0 \end{bmatrix}$$

$$\Delta_B = b_1 \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} + b_2 \begin{bmatrix} -0.6 \\ 0.5 \end{bmatrix}, \quad \Delta_C = c_1 \begin{bmatrix} 1.6 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \end{bmatrix}$$

where $a_1, a_2 \in [-0.02, 0.02]$, $b_1, b_2 \in [-0.1, 0.1]$ and $c_1, c_2 \in [-0.1, 0.1]$, $\Delta_B = 0$.

Results of frequency domain LTV analysis using proposed method are shown on fig. 1 and 2. Magnitude diagrams for LTV system with $r=0.1$ are depicted on figure 1. Nominal system without perturbations is depicted by solid line. Upper bound estimated using the method described in former section is drawn by dashed line. Second example with $r=1$ is depicted on figure 2. Nominal system is plotted by solid line, while upper bound is denoted by dashed line.

Classical worst case gain analysis implemented for LTI systems, cannot be applied directly to LTV systems. Neglected time variability with connection to classical LTI system worst case gain analysis follows to false results. The bounds may be either overestimated or underestimated. When time-variability is converted into uncertainty the bounds are overestimated. Estimates are conservative and in fact useless. Our approach allows to
estimate upper bounds for LTV systems, while results for LTI systems or systems with negligible variability are close to classical worst case gain methods.

The method enable worst case gain analysis for uncertain time-varying systems what was not possible previously. However it should be also noted some imperfections of them. First of all approximated Bode diagrams are significant simplification of the LTV system. Such simplification may be either advantage or disadvantage of the method, dependent on the specific application. In general it follows to approximate analysis and approximate worst case gain evaluation. Second limitation is connected to the condition of the matrix Taylor series convergence, equivalent to \( \| \hat{L}_\Delta \| \ll 1 \) and simplifying assumption \( (\hat{L}_\Delta)^n = 0 \) for \( n=2,3,... \). In practise it means that maximal perturbation \( \Delta_L \) which may be considered is bounded by norm of inverse to the operator \( \hat{L} \).

**REFERENCES**


