The Decoupling of a Harmonic-Drive-Spring System for Position and Torque Control on Two Different Axes

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Abstract—The motivation of this paper is to implement a control algorithm to control the angular position of one axis and the torque on another axis in a harmonic drive system. Obviously, two independently operated actuators on any two axes form an essential part of the setup. One of the motors is coupled through a spring-damper element to the corresponding shaft. The paper summarizes the most important properties of harmonic drives. It gives the torque equations of the harmonic-drive-spring system and also presents a minimal state-space realization for controller synthesis. The decoupling of the multivariable system by state feedback with symbolic calculations is achieved.

I. INTRODUCTION

The concept of the harmonic drive was patented by C.W. Musser in 1955 [1]. This new gear concept used to be applied in aerospace and other specific applications but today it is used in many fields such as robotics, machine tools, medical equipment and automotive industry.

Harmonic drives have high speed reduction and torque multiplication ratios using single stage and coaxial configuration of shafts. Other benefits include nearly zero backlash, small size and high torque transmission capacity due to the high number of teeth in contact.

The harmonic drive is made up of three basic components: the wave generator, the flexspline and the circular spline as depicted in Fig. 1. The wave generator is an elliptical cam enclosed in an antifriction ballbearing assembly. It is inserted into the bore of the externally toothed flexspline. There are 2 less teeth on the flexspline than on the internally toothed circular spline. The flexspline is deformable and takes on the elliptical shape of the wave generator causing its external teeth to engage with the internal teeth of the circular spline at two opposite points. Thanks to this mechanism, the number of contacted teethes in harmonic drives is greater than in traditional gears.

Rotating the wave generator causes rotation of the flexspline relative to the circular spline in an opposite direction due to the difference in the number of teeth. Every harmonic drive is assigned to a transmission ratio denoted by $N$ that describes how many revolutions of the wave generator causes one revolution in the flexspline relative to the circular spline.

Currently, harmonic drives in most applications are applied to provide high efficiency gearing without using complex mechanisms and structures. Such driving configurations can be obtained if one of the three axes is fixed. Usually the circular spline is fixed and the input is through the wave generator while the output is via the flexspline that rotates $N$ times slower as the wave generator in the opposite direction relative to the circular spline. In a configuration where the flexspline is held stationary and the transmission of motion is from the wave generator to the circular spline, the circular spline rotates $N+1$ times slower as the wave generator in the same direction relative to the flexspline. Both of these driving configurations belong to the reduction gearing configurations. The modeling, identification and control of harmonic drives in reduction gearing configuration have a rich literature such as [2], [3], [4] including the consideration of the non-linear compliance and torque transmission due to the non-rigid behavior of the flexspline. The related SISO position control problem is also addressed in several papers, e.g. in [5]. The authors of [6] report a robust controller synthesis approach whereas adaptive control is suggested in [7].

This paper considers an alternative configuration i.e. a differential gearing configuration of harmonic drives, where none of the shafts is fixed, like in [8]. Nevertheless, the study in [8] dealt with a system where the inertias were connected directly to the harmonic drive shafts but...
II. MODELING

Now we give the differential equations, which govern the motion of the axes. In order to obtain linear model, we suppose viscous friction to be the only damping phenomena and a linear spring behavior.

The relation constraining the motion of the harmonic drive shafts is given as

\[ \varphi_{wg} - (N + 1) \varphi_{cs} + N \varphi_{fs} + \varphi_0 = 0 \]  

(1)

where the constant \( \varphi_0 \) can introduce offset to the angular positions, e.g. in the case of relative angle measurement devices such as optical encoders. We will suppose \( \varphi_0 = 0 \) that can be assured at the initialization of the sensor devices.

Notice that (1) may also be interpreted as restricting the motion to a subspace of the vector space spanned by the shaft angles \((\varphi_{wg}, \varphi_{cs}, \varphi_{fs})\), namely to the plane orthogonal to the vector

\[ c = \begin{pmatrix} 1 \\ -(N + 1) \end{pmatrix} \]  

(2)

A. Equations of Motion

The motion of the system is governed by the Lagrange equations (3) that are obtained using the Lagrange’s-’Alambert principle.

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F \]  

(3)

Here \( q \) stand for the vector of the generalized coordinates. The vector \( q = (\varphi_{wg}, \varphi_{SDmot}, \varphi_{fs})^{T} \) is a proper choice for the vector of generalized coordinates because the three variables \( \varphi_{wg}, \varphi_{SDmot} \) and \( \varphi_{fs} \) specify the position of all points of the system.

\( L \) is the Lagrangian, which is given as the difference of the kinetic and potential energy:

\[ L = K - P \]  

(4)

The kinetic energy \( K \) is a quadratic form with respect to the generalized velocities \( \dot{q} \):

\[ K = \frac{1}{2} \dot{q}^T H \dot{q} \]  

(5)

where the inertia matrix reads

\[
H = \begin{bmatrix}
J_{wg} & 0 & 0 \\
0 & J_{SDmot} & 0 \\
0 & 0 & J_{fs}
\end{bmatrix}
\]  

(6)

The potential energy is stored in the spring:

\[ P = \frac{1}{2} S_{SD} \Delta \varphi^2 \]  

(7)

where \( \Delta \varphi = \varphi_{cs} - \varphi_{SDmot} \) is the angular displacement between the ends of the spring-damper element. After substitution of the explicit expression of \( \varphi_{cs} \) obtained form (1) this angular displacement reads

\[ \Delta \varphi = \frac{1}{N + 1} (\varphi_{wg} - (N + 1) \varphi_{SDmot} + N \varphi_{fs}) = \frac{1}{N + 1} c^T q \]  

(8)

where \( c \) is the vector defined in (2). Hence the potential energy is a quadratic form with respect to the generalized positions \( q \):

\[ P = \frac{1}{2} q^T S q \]  

(9)

where

\[ S = \frac{S_{SD} c c^T}{(N + 1)^2} \]  

(10)

\( F \) is the vector of the generalized external forces. The generalized external forces include the external torques.
with help of Rayleigh’s dissipative function that is defined as a quadratic form with respect to the generalized velocities $\dot{q}$:

$$R = \frac{1}{2} \dot{q}^T D \dot{q}$$  \hspace{1cm} (11)$$

The matrix $D$ is symmetric and positive definite. Then $F$ reads

$$F = T - \frac{\partial R}{\partial \dot{q}} = T - D \dot{q}$$  \hspace{1cm} (12)$$

The dissipative torque $-D \dot{q}$ describes the viscous friction. In our case, $D$ is the sum of two matrices. The first term represents the damping on the axes $w_g$, $SD_{mot}$ and $f_s$ with the viscous friction coefficients $b_{wg}$, $b_{SD_{mot}}$ and $b_{fs}$ in the diagonal. The second term represents the damping of the spring-damper element and its derivation is analogous to the derivation of $S$ in (9) with respect to the corresponding term in Rayleigh’s dissipative function:

$$R_{SD} = \frac{1}{2} b_{SD} \Delta \dot{\varphi}^2$$  \hspace{1cm} (13)$$

The only difference in the derivation is that we have to substitute the generalized velocities $\dot{q}$ and the damping factor $b_{SD}$ instead of the generalized positions $q$ and the stiffness coefficient of the spring $S_{SD}$, respectively. The result for $D$ has the following form:

$$D = \begin{bmatrix} b_{wg} & 0 & 0 \\ 0 & b_{SD_{mot}} & 0 \\ 0 & 0 & b_{fs} \end{bmatrix} + \frac{b_{SD}}{(N + 1)^2} \varphi_T$$  \hspace{1cm} (14)$$

We have determined all coefficients to give the Lagrange equations

$$H \ddot{q} + D \dot{q} + S \dot{q} = T$$  \hspace{1cm} (15)$$

B. State-Space Realization

Equation (15) can be transformed into a system of first order differential equations

$$\dot{x} = Ax + Bu$$  \hspace{1cm} (16)$$

with $x = (q, \dot{q})^T$ and

$$A = \begin{bmatrix} 0 & I \\ -H^{-1}S & -H^{-1}D \end{bmatrix}$$  \hspace{1cm} (17)$$

The actuators are mounted to the wave generator shaft and to the circular spline shaft after the spring-damper. Accordingly, the input is $u = (T_{wg}, T_{SD_{mot}})^T$ and $B$ reads

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$  \hspace{1cm} (18)$$

Nonetheless, observe that the first three columns of matrix $A$ are linearly dependent because the matrix $S$ is of rank 1. It means that in order to express $\dot{q}$ we do not need to know the values of variables $\dot{\varphi}_{wg}$, $\dot{\varphi}_{SD_{mot}}$ and $\dot{\varphi}_{fs}$ but only their linear combination i.e. the value of $c^T \dot{q}$. Moreover, the system outputs, i.e. the controlled variables are the angular position of the circular spline shaft $\varphi_{cs}$ and the torque on the flexspline shaft, so the variables $\varphi_{wg}$, $\varphi_{fs}$ are not required and they can be replaced with $c^T \dot{q}$. This replacement reduces the order of the state-space model from 6 to 5. In this reduced order model, the vector of the state-variables is $x = (\varphi_{SD_{mot}}, c^T \dot{q})^T$ and the matrix $A$ reads

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -b_{fs} & 0 \end{bmatrix}$$  \hspace{1cm} (19)$$

To obtain the matrix $B$ for the reduced order model, we have to leave one of the first three zero rows out of $B$ in (18), i.e. the dimension of the zero matrix in the first factor is reduced from $3 \times 3$ to $2 \times 3$.

To calculate the second output in knowledge of the state variable $x$, we refer to the well-known Newton’s law:

$$T_{fs} = J_{fs} \ddot{\varphi}_{fs}$$  \hspace{1cm} (20)$$

Notice that the angular acceleration $\ddot{\varphi}_{fs}$ of the flexspline shaft is expressed by the last equation in (16). After substitution of (2), (6) and (14) into (19) the output equations read

$$y = Cx$$  \hspace{1cm} (21)$$

where

$$C = \begin{bmatrix} 1 & 0 & -N b_{SD} (N + 1)^2 & -N b_{fs} (N + 1) & 0 \\ 0 & -N b_{SD} & 0 & -b_{fs} & 0 \end{bmatrix}$$  \hspace{1cm} (22)$$

To check the minimality we use the theorem that a realization $(A, B, C)$ is minimal if and only if it is controllable and observable [11]. Both the controllability and the observability properties can be proven easily and minimality of the reduced order model with model order 5 is shown.

III. Decoupling

We will use the comprehensive theory introduced in [9] and the synthesis procedure described in [10] in order to decouple the multivariable system given by (16) and (21) with matrices $A$, $B$ and $C$ from (19), (18) and (22), respectively, by state feedback. We assume the measurement of the state-variables that means the measurement of angular position and velocity signals. The key to the solution of the decoupling problem is a canonical representation of integrator decoupled systems. The basic steps of the controller synthesis are the inspection of the necessary and sufficient conditions for decoupling, the conversion to integrator decoupled form, the transformation to a canonically decoupled form and the compensation of the canonically decoupled system.

A. Necessary and Sufficient Conditions for Decoupling

Let us consider the $m$-input, $n$-output, $n$th order system $(A, B, C)$, where $A$, $B$, $C$ are respectively matrices of size $n \times n$, $n \times m$, $m \times n$. We introduce the integers $r_i$ (elements of the vector relative degree) and the $1 \times m$ row matrices $G_i (i = 1, \ldots, m)$ as follows:

$$r_i = \begin{cases} n, & \text{if } C_i A^k B = 0, \forall k = 0, 1, \ldots, n-1 \\ j + 1, & \text{if } C_i A^k B = 0, \forall k = 0, 1, \ldots, j-1 \text{ and } C_i A^j B \neq 0 \end{cases}$$  \hspace{1cm} (23)$$
where $C_i$ is the $i$th row of $C$. We form the $m \times m$ matrix

$$G = \begin{bmatrix} G_1 \\ \vdots \\ G_m \end{bmatrix} \quad (25)$$

The system can be decoupled if and only if the so-called decoupling matrix $G$ is nonsingular.

For our system $r_1 = 2$, $r_2 = 1$ and

$$G = \begin{bmatrix} 0 & \frac{1}{N} J_{bgmot} \\ \frac{1}{N+1} J_{bdmot} & 0 \end{bmatrix} \quad (26)$$

It is easy to examine that the determinant $|G| \neq 0$ i.e. $G$ is nonsingular and the system can be decoupled. For future use we also calculate its inverse

$$G^{-1} = \begin{bmatrix} (N+1) J_{bg} & -\frac{(N+1)^2}{N} b_{bgmot} \\ \frac{N J_{bdmot}}{b_{bgmot}} & (N+1) J_{bd} \end{bmatrix} \quad (27)$$

B. Conversion to Integrator Decoupled Form

The system $(A, B, C)$ is integrator decoupled if the transfer matrix $H(s) = C(I_n s - A)^{-1} B$ is diagonal and has diagonal elements

$$h_i(s) = \frac{\gamma_i}{s r_i}, \quad (i = 1, \ldots, m) \quad (28)$$

Consider the system $(A, B, C)$, where $G$ is nonsingular and let $A^*$ denote the $m \times n$ matrix

$$A^* = \begin{bmatrix} C_1 A'^1 \\ \vdots \\ C_m A'^m \end{bmatrix} \quad (29)$$

Then with the control law

$$u = -G^{-1} A^* x + G^{-1} v \quad (30)$$

the closed-loop system (31) is integrator decoupled.

$$(A - B G^{-1} A^*, B G^{-1}, C) \quad (31)$$

In our case $G^{-1}$ is as in (27) and the expression of $A^*$ reads

$$A^* = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\frac{S_{bgmot}}{(N+1)^2} & \frac{S_{bdmot}}{b_{bgmot}} & \cdots & 0 \\ 1 & -b_{bgmot} & \cdots & 0 \\ 0 & -b_{fgmot} & \cdots & 0 \end{bmatrix} T A \quad (32)$$

C. Transformation to Canonically Decoupled Form

The system $(A, B, C)$ is canonically decoupled if the following conditions are satisfied:

1) The matrices $A$, $B$ and $C$ have the partitioned form:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 & A_1^u \\ 0 & A_2 & \cdots & 0 & 0 & A_2^u \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m & 0 & A_m^u \\ A_1^c & A_2^c & \cdots & A_m^c & A_{m+1} & A_{m+1}^u \\ 0 & 0 & \cdots & 0 & 0 & A_{m+2}^u \end{bmatrix} \quad (33)$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m+1} \\ b_1^u \\ \vdots \\ b_m^u \end{bmatrix} \quad (34)$$

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \\ c_1^u \\ \vdots \\ c_m^u \end{bmatrix} \quad (35)$$

2) For $i = 1, \ldots, m$ the matrices $A_i, b_i$ and $c_i$ have the partitioned form:

$$A_i = \begin{bmatrix} 0 & I_{r_i - 1} & 0 \\ 0 & \Upsilon_i & \Phi_i \end{bmatrix} \quad (36)$$

$$b_i = \begin{bmatrix} \frac{\gamma_i}{r_i} \\ \beta_i \\ \beta_{di} \end{bmatrix} \quad (37)$$

$$c_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \quad (38)$$

Dimensions of the matrix partitions are given in Table II where $d_i$ is defined as $d_i = p_i - r_i$.

**TABLE II**

**DIMENSIONS OF MATRIX PARTITIONS**

<table>
<thead>
<tr>
<th>$A_i$</th>
<th>$B_i$</th>
<th>$C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1 \times p_1$</td>
<td>$p_1 \times 1$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$p_m+1 \times p_1$</td>
<td>$p_{m+1} \times 1$</td>
<td>$1 \times p_m$</td>
</tr>
<tr>
<td>$d_i \times r_i$</td>
<td>$1 \times p_{m+2}$</td>
<td>$p_i \times d_i$</td>
</tr>
</tbody>
</table>

3) For $i = 1, \ldots, m$ the systems $(A_i, b_i, c_i)$ are controllable.

4) Let $\eta = [\eta_1 \eta_2 \cdots \eta_{m+2}]$ be a partitioned $n$ row where $\eta_i$ is a $p_i$ row. If $p_{m+1} = d_{m+1} > 0$ and $\eta_{m+1} \neq 0$, then the $1 \times m$ row matrix function $\eta(I_n s - A)^{-1} B$ has at least two nonzero elements.

For a canonically decoupled system the decoupling problem has a particularly simple form and Gilbert shows, in addition, that every integrator decoupled system is similar to a canonically decoupled system [9]. The transformation is the product of two transformations. The first transformation separates the controllable and uncontrollable modes. As our system is controllable we omit this first transformation and the uncontrollable mode. The second transformation affects only the controllable mode and the transformation matrix is

$$Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_m \\ Q_{m+1} \end{bmatrix} \quad (39)$$

To determine $Q$, the following notation is required. Let $M_i$ be the linear row space $\{ \eta : \eta A^b B_k = 0, \forall k \in \{1, \ldots, m\}, \forall j \in \{0, \ldots, n - 1\} \}$ where $B_k$ is
The transpose of these basis columns. The rows of $L$ are independent, and consequently $\text{rank}(L) = n$. Recall that $k$ is the controllability matrix. Accordingly, $Q_i = C_i A^{r_i - 1}$ (40) where $L_i = [P_1 \cdots P_{i-1} P_{i+1} \cdots P_n]$ and $P_k = [B_k \cdots A^{n-k} B_k]$ is the controllability matrix of the pair $(A, B_k)$. The rows $q_i^1, \ldots, q_i^{p_i - r_i}$, $(i = 1, \ldots, m)$, can be formed by taking the transpose of these basis columns. This is done by calculating a basis for the column null space of $[Q_1^T \cdots Q_m^T]$. The rows of $Q_{m+1}$ must extend the rows of $Q_1, \ldots, Q_m$ to form a set of $n$ linearly independent rows. This is done by calculating a basis for the column null space of $[Q_1^T \cdots Q_m^T]$.

The $q_i^1$ can be determined by calculating a basis for the column null space of $[L_i \ C_i^T \cdots (C_i A^{d_i})^T]^T$ where $L_i = [P_1 \cdots P_{i-1} P_{i+1} \cdots P_n]$ and $P_k = [B_k \cdots A^{n-k} B_k]$ is the controllability matrix of the pair $(A, B_k)$. $B_k$ is the $k$th column of $B$. The rows $q_i^1, \ldots, q_i^{p_i - r_i}$, $(i = 1, \ldots, m)$, are zero and the third and fourth rows are linearly independent. Moreover the $M_i$ are disjoint so that it is possible to write $M = M_1 \oplus M_2 \oplus \cdots \oplus M_{m+1}$. The rows of $Q_i$ are a basis for $M_i$.

$$Q_i = \begin{bmatrix} C_i \\ C_i A^{r_i - 1} \\ \vdots \\ q_i^1 \\ \vdots \\ q_i^{p_i - r_i} \end{bmatrix}, \quad (i = 1, \ldots, m) \quad (40)$$

$$Q_{m+1} = \begin{bmatrix} q_{m+1}^1 \\ \vdots \\ q_{m+1}^{p_{m+1}} \end{bmatrix} \quad (41)$$

The $q_i^1$ can be determined by calculating a basis for the column null space of $[L_i \ C_i^T \cdots (C_i A^{d_i})^T]^T$ where $L_i = [P_1 \cdots P_{i-1} P_{i+1} \cdots P_n]$ and $P_k = [B_k \cdots A^{n-k} B_k]$ is the controllability matrix of the pair $(A, B_k)$. The rows $q_i^1, \ldots, q_i^{p_i - r_i}$, $(i = 1, \ldots, m)$, can be formed by taking the transpose of these basis columns. This is done by calculating a basis for the column null space of $[Q_1^T \cdots Q_m^T]$. The rows of $Q_{m+1}$ must extend the rows of $Q_1, \ldots, Q_m$ to form a set of $n$ linearly independent rows. This is done by calculating a basis for the column null space of $[Q_1^T \cdots Q_m^T]$. The rows $q_{m+1}^1, \ldots, q_{m+1}^{p_{m+1}}$ are the transpose of the resulting basis columns. It is obvious that

$$p_i = \max(n - \text{rank}(L_i), r_i), \quad (i = 1, \ldots, m) \quad (42)$$

$$p_{m+1} = n - \sum_{i=1}^{m} p_i \quad (43)$$

Let us transform the system given by (31) to canonically decoupled form. The elements of matrices $L_i$ are very sensitive to the system parameters. Their ranks and the above mentioned null space cannot be reliably determined with numerical methods and we have to apply symbolic computations. After lengthy but straightforward symbolic calculations we obtain that the first and fourth rows of $L_1$ are zero and the other 3 rows are linearly independent, and consequently $\text{rank}(L_1) = 3$. Substituting this and $r_1 = 2$ into (42) we obtain $p_1 = 2$. The rank of $L_2$ turns out to be 2 because its second and fifth rows are zero and the third and fourth rows are linearly dependent. Recall that $r_2 = 1$ and we obtain $p_2 = 3$, $p_3 = 0$. Accordingly, $Q_1$ and $Q_2$ read

$$Q_1 = \begin{bmatrix} C_1 \\ C_1 (A - B G^{-1} A^*) \end{bmatrix}, \quad Q_2 = \begin{bmatrix} C_2 \\ q_2^1 \\ q_2^2 \end{bmatrix} \quad (44)$$

Because the second and fifth rows of $L_2$ are zero the second and fifth standard unit vectors $e_2^T$ and $e_5^T$ are appropriate for $q_2^1$ and $q_2^2$. Accordingly, $Q$ and its inverse read

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{N_{bd} S_{bd}}{N_{bd} S_{bd} + b_s - N_{bd}} & -\frac{S_{bd} S_{bd}}{N_{bd} S_{bd} + b_s - N_{bd}} & -\frac{(N^2 s^2)_{bd}}{N_{bd} S_{bd} + b_s - N_{bd}} & -\frac{(N^2 s^2)_{bd}}{N_{bd} S_{bd} + b_s - N_{bd}} \end{bmatrix} \quad (45)$$

$$Q^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (46)$$

The canonically decoupled form $(A_{cd}, B_{cd}, C_{cd})$ of the integrator decoupled system (31) is given by

$$A_{cd} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_{cd} = Q B G^{-1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (48)$$

$$C_{cd} = C Q^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (49)$$

### D. Compensations of the Canonically Decoupled System

The canonically decoupled system has two partitions, namely $(A_1, b_1, c_1)$ of order 2 and $(A_2, b_2, c_2)$ of order 3. Equation (28) suggests, however, that $(A_1, b_1, c_1)$ is the realization of a double-integrator and the second output is the integral of the second input. The first statement can easily be detected considering the realization: its two poles i.e. the roots of the characteristic polynomial $\lambda^2 = 0$ are both zero and the gain equals 1. For the second partition, notice that the derivative of the state-variable, which state-variable also is the output, equals the input. The other two state-variable evolve depending on this output variable but have no effect on the input-output behavior. This pole-zero cancellation will be treated in more detail later on.

1) Compensation of the First Partition: The compensation of partition $(A_1, b_1, c_1)$ is equivalent to the compensation of the double integrator:

$$\dot{\tilde{y}}_1 = v_1 \quad (50)$$

Suppose that we want to achieve a second-order closed-loop behavior:

$$\dot{\tilde{y}}_1 + 2 \xi \omega_1 \tilde{y}_1 + \omega_1^2 \tilde{y}_1 = \omega_1^2 \tilde{y}_{ref,1} \quad (51)$$

where $\omega_1$ is the natural undamped frequency and $\xi$ is the damping factor. From (50) and (51) we obtain:

$$v_1 = \omega_1^2 \tilde{y}_{ref,1} - 2 \xi \omega_1 \hat{y}_1 - \omega_1^2 \hat{y}_1 \quad (52)$$
Computation of the Second Partition: The eigenvalues of the $3 	imes 3$ partition $A_2$ are located at $0$, $0$, $-S_{SD}/b_{SD}$. The invariant zeros are the solution of

$$
\begin{align*}
    sI - A_2 &- b_2 \\
c_2 &0
\end{align*}
= s \left( s + \frac{S_{SD}}{b_{SD}} \right) = 0 \tag{53}
$$

The roots are $0$ and $-S_{SD}/b_{SD}$ that cancel the corresponding poles and $(A_2, b_2, c_2)$ turns out to be an integrator. The cancelled poles correspond to the zero dynamics of partition 2, however, and should normally be stable. According to (47)-(49), the first state variable in this partition is the second output i.e. $T_f$, and the last state variable that introduced the unstable mode at $0$ is $\int T_f$. Recall (20) and recognize that this state variable equals the angular velocity of the flexspline axis. We want to control the torque that accelerates the flexspline, and consequently we cannot avoid this mode. On the contrary, this mode in the zero dynamics is even needed to achieve the torque control. Therefore, the control law reads

$$
v_2 = \omega_2 (y_{ref,2} - y_2) \tag{55}
$$

Equations (52) and (55) can be summarized as

$$
v = -K_{cd}x_{cd} + L_{cd}y_{ref} \tag{56}
$$

where $x_{cd}, v = (v_1, v_2)^T, y_{ref} = (y_{ref,1}, y_{ref,2})^T$ are the vectors of the state-variables, the vector of the inputs of the canonically decoupled system and the vector of the references, respectively. The coefficient matrices are

$$
K_{cd} = \begin{bmatrix}
\omega_1^2 & 2\omega_1 \omega_2 & 0 & 0 \\
0 & \omega_2 & 0 & 0
\end{bmatrix} \tag{57}
$$

$$
L_{cd} = \begin{bmatrix}
\omega_1^2 & 0 \\
0 & \omega_2
\end{bmatrix} \tag{58}
$$

E. Closed-Loop System

We substitute (56) into (30) and consider the transformation $x_{cd} = Qx$. For the the control input we obtain

$$
u = -Kx + Ly_{ref} \tag{59}
$$

where $K = G^{-1}(A^* + K_{cd}Q)$ and $L = G^{-1}L_{cd}$.

IV. CONCLUSION

We achieved the decoupling of the harmonic-drive-spring system to control position and torque on two different axes. The main feature of the paper is the symbolic derivation because the problem is numerically ill-conditioned.

To prove the results, Figure 3 presents step responses of the closed-loop $(A - BK, BL, C)$ with parameter values $N = 50$, $J_{wg} = 10^{-4}$, $J_{SD mot} = J_f = 0.05$, $b_{wg} = 5 \cdot 10^{-4}$, $b_{SD} = b_f = b_{SD mot} = 0.01$, $S_{SD} = 100$, $\omega_1 = \omega_2 = 20$, $\xi = 0.7$. Note that control law (59) decouples $(A, B, C)$ with the desired performance.

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