Geometric control in a regulator problem for electrohydraulic servos

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Abstract – A five-dimensional nonlinear mathematical model of the electrohydraulic servo(mechanism) is considered. In the system equilibria analysis, the critical case of a zero eigenvalue occurs. The Lyapunov–Malkin Theorem and Routh-Hurwitz criterion provide conditions for controllers to stabilize all relevant equilibria in the closed-loop system. Geometric control paradigm is then applied in synthesis and the performance of the obtained controlled system is numerically validated from viewpoint of the regulator classical problem.

I. INTRODUCTION

The present paper addresses the problem of a stabilizing – regulator type – geometric control for an electrohydraulic servo(mechanism) (EHS). The classical approach in control design for a given EHS mathematical model was to start by linearising the nonlinear dynamics around a specific equilibrium point and use linear design methodology to obtain the control law (see the pioneering books in the field [1]-[4]). The modern approach in the synthesis of control laws for EHSs is the development of strategies directly applicable to large classes of nonlinear models: optimal nonlinear control, adaptive robust control, backstepping control, neuro-fuzzy control, sliding mode control [5-14]. Among the most recent techniques applied in the fields one finds those in [15] and [16], based on differential geometric methods [17]. It is in this setting that in the present paper a controller is provided for an aviation EHS, whose mathematical model is described by the following systems of ordinary differential equations (see also [18], [19], and Fig. 1); for $x_5 \geq 0$

$$\begin{align*}
\dot{x}_1 &= x_2; \\
\dot{x}_2 &= (-k_1 - f_x x_2 + S x_3 - S x_4) / m; \\
\dot{x}_3 &= -\frac{1}{\tau} x_5 + \frac{k}{\tau} u_1; \\
\dot{x}_4 &= \frac{B}{V_0 + S x_1} \left( C x_5 \sqrt{P_2 - x_2} \right); \\
\dot{x}_5 &= \frac{B}{V_0 - S x_1} \left( C x_5 \sqrt{P_1 - x_2} \right)
\end{align*}
$$

(1.1)

and, for $x_5 \leq 0$,

$$\begin{align*}
\dot{x}_1 &= x_2; \\
\dot{x}_2 &= (-k_1 - f_x x_2 + S x_3 - S x_4) / m; \\
\dot{x}_3 &= -\frac{1}{\tau} x_5 + \frac{k}{\tau} u_2; \\
\dot{x}_4 &= \frac{B}{V_0 + S x_1} \left( C x_5 \sqrt{P_2 - x_2} \right); \\
\dot{x}_5 &= \frac{B}{V_0 - S x_1} \left( C x_5 \sqrt{P_1 - x_2} \right)
\end{align*}
$$

(1.2)

Figure 1. Physical model of EHS including: HC – hydrocylinder; EHSV – electrohydraulic servovalve; L – load; T – transducers system; µP – microprocessor; TM – torque motor

The characteristic features of the associated physical model in Fig. 1 are: a double-ended actuator, equal ram areas of the two chambers, well known and constant load inertia, negligible dry friction forces in the ram. In (1.1), (1.2), $x_1$ is the load displacement, $x_2$ is the load velocity, $x_3$ and $x_4$ are the pressures in the cylinder chambers, $x_5$ is the valve position and $u_1$, $u_2$ are the control variables (input voltages). The constants involved are: $m$, the equivalent inertial load of primary control surface reduced to the actuator rod; $f_x$, an equivalent viscous friction force coefficient; $k$, an equivalent aerodynamic elastic force coefficient.
coefficient; \( S \), the effective area of the piston; \( V_0 \), the cylinder half-volume; \( p_s \), the supply pressure; \( B \), the bulk modulus of oil; \( \tau \), the servovalve time-constant; \( c_d \), volumetric flow coefficient of the valve port; \( w \), the valve port’s width; \( \rho \), the oil volumetric density; \( k_v \), a proportionality coefficient relating the input voltage to servovalve to valve displacement.

In what follows, we regard (1.1) and (1.2) as independent control systems where the variation of state variables is limited only by the algebraic conditions that the square roots are well defined; these conditions correspond in fact to noncavitating physical conditions \( 0 < x_i < p_s, i = 3, 4 \).

Let the reference signal \( x_0 \) (in Volts, [V]) be scaled so that \( |x_0| < \frac{V_0}{S} \). For \( p \in (0, 1) \) that satisfies the conditions

\[
p_s(1 - p) - kx_0 / S > 0, \text{if } x_3 > 0, \text{ or } \]
\[
p_p + kx_0 / S > 0, \text{if } x_3 > 0 \tag{1.3}
\]

define

\[
\begin{align*}
\dot{x}_1 &= x_0, \\
\dot{x}_2 &= 0, \\
\dot{x}_3 &= \frac{kx_0}{S} + p_p x_5, \\
\dot{x}_4 &= p_s x_5, \\
\dot{x}_5 &= 0
\end{align*}
\tag{1.4}
\]

If \( u_1(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5) = 0 \) and \( u_2(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5) = 0 \), (1.4) are equilibrium points both for the systems (1.1) and (1.2). Thus, the conditions (1.3) ensures that the square roots are well defined. As it will be shown in Section 2, the Jacobian matrices of both systems (1.1) and (1.2), calculated in equilibria (1.4), have zero in spectrum, so we have to face a critical case for stability theory (also a critical case in control synthesis, see [17]). This case is handled through the use of a theorem of Lyapunov and Malkin (see [20]) and, consequently, in the work, a general criterion that a stabilizing controller must fulfil is used.

In the present work, the EHS tracking system [11], [13] is considered from the special viewpoint of the regulator problem [21]. The qualitative statement of this problem is the following: Suppose initial conditions for some solution of the system (1.1) or (1.2) are different from equilibria. Provide a control law \( u_1(\ell), u_2(\ell) \), to bring this solution to an equilibrium point. Theorem 2.1 in Section 2 gives a solution to this problem, giving conditions that a control law performs this task with a certain error.

The paper is organized as follows. In Section 2 is presented a general criterion (Theorem 2.1) for controllers \( u_1, u_2 \) to stabilize equilibria (1.4) for (1.1) and (1.2), respectively. In Section 3 geometric methods are used to provide feedback control laws \( u_1 \) and \( u_2 \) that eventually fulfill the general criterion deduced in Section 2 for all equilibria (1.4). It should be mentioned that \( u_1 \) and \( u_2 \) have an explicit analytic dependence of equilibria, a major progress from construction in [22]. Section 4 succinctly presents numerical calculations and simulations to validate theoretical findings. A final Section 5 is devoted to conclusions.

### II. STABILITY ANALYSIS

The stability of equilibria (1.4) will be studied only for the system (1.1), similar reasoning being applicable to (1.2). When these equilibria are translated into zero by

\[
\begin{align*}
y_1 &= x_1 - \hat{x}_1, \\
y_2 &= x_2, \\
y_3 &= x_3 - \hat{x}_3 \\
y_4 &= x_4 - \hat{x}_4, \\
y_5 &= x_5
\end{align*}
\tag{2.1}
\]

and \( \tilde{u} \) is defined by

\[
\tilde{u}(\tilde{y}) = \frac{k_v}{\tau} \tilde{u}(\tilde{y} + \hat{x})
\tag{2.2}
\]

system (1.1) becomes

\[
\begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -\frac{k}{m} y_1 - \frac{f_x}{m} y_1 - \frac{f_v}{m} y_2 - \frac{S}{m} y_3 - \frac{S}{m} y_4 + \frac{BC}{V_0 + S y_1 + S x_0} y_5 \sqrt{y_4 + y_4} + \frac{BS y_2}{V_0 - S y_1 - S x_0} y_5 \sqrt{y_4 + y_4} + \frac{BS y_3}{V_0 - S y_1 - S x_0} y_5 \\
\dot{y}_3 &= \frac{BC}{V_0 + S y_1 + S x_0} y_5 \sqrt{y_4 + y_4} + \frac{BS y_2}{V_0 - S y_1 - S x_0} y_5 \sqrt{y_4 + y_4} + \frac{BS y_3}{V_0 - S y_1 - S x_0} y_5 \\
\dot{y}_4 &= \frac{BC}{V_0 + S y_1 + S x_0} y_5 \sqrt{y_4 + y_4} + \frac{BS y_2}{V_0 - S y_1 - S x_0} y_5 \sqrt{y_4 + y_4} + \frac{BS y_3}{V_0 - S y_1 - S x_0} y_5 \\
\dot{y}_5 &= \frac{1}{\tau} y_5 + \tilde{u}(\tilde{y})
\end{align*}
\tag{2.3}
\]

Denote by \( A \) the Jacobian matrix in zero

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
-\frac{k}{m} - \frac{f_x}{m} & -\frac{S}{m} & -\frac{S}{m} & 0 \\
0 & a_{32} & 0 & 0 & a_{35} \\
0 & a_{42} & 0 & 0 & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & -\frac{1}{\tau}
\end{pmatrix}
\tag{2.4}
\]

where

\[
a_{32} = -\frac{BS}{V_0 + S x_0}, \quad a_{35} = \frac{BC}{V_0 + S x_0} \left[ p_s(1 - p) - \frac{kx_0}{S} \right],
\]

\[
a_{51} = \frac{\partial \tilde{u}}{\partial y_1}(0), \quad a_{52} = \frac{\partial \tilde{u}}{\partial y_2}(0), \quad a_{53} = \frac{\partial \tilde{u}}{\partial y_3}(0), \quad a_{54} = \frac{\partial \tilde{u}}{\partial y_4}(0).
\]
\[ a_{42} = \frac{BS}{V_0 - Sx_0}, \quad a_{45} = -\frac{BC \sqrt{p_1 p}}{V_0 - Sx_0}. \]

It is easy to see that the characteristic polynomial \( Q(\lambda) \) of \( A \) has zero as a root, so

\[ Q(\lambda) = \lambda Q_1(\lambda) \quad (2.5) \]

This corresponds to a critical case for stability theory that can be approached through the use of a theorem of Lyapunov and Malkin [20].

The following theorem addresses the problem of stability for the zero solution of the system (2.3).

**Theorem 2.1** [23]. Suppose \( Q_1 \) in (2.5) is a stable polynomial (all its roots have strictly negative real parts). Then the zero solution of (2.3) is simply stable by Lyapunov. If initial perturbations of an equilibrium point are small enough, \( \lim_{t \to \infty} y_i(t) \) can be made arbitrary small, thus \( \lim_{t \to \infty} x_i(t) \to \tilde{x} \) is arbitrary small; in other words, any equilibrium \( |x_0| < \frac{V_0}{S} \) can be recovered by the output \( x_i \) as close it is necessary.

In fact, this result represents a solution for the regulator type problem of EHS. From reasons of limited printing space, the proof is herein omitted. We only note that the use of Lyapunov-Malkin Theorem requires several variables changes of the original system that finally brings it to a specific form where the reasoning of that theorem can be applied. See also [18] for details.

### III. GEOMETRIC CONSTRUCTION

General references for geometric control theory to be used in the sequel are [24] and [25]. For the specific use of these methods in the field of tracking systems, see [16].

Considering the system (1.1) and applying the machinery of the geometric control construction, an intermediary nonlinear function of states is obtained [23]

\[ h_1(x_1, x_2, x_3, x_4) = (V_0 + Sx_1) \sqrt{p_1 - x_3} - \]

\[ -(V_0 - Sx_1) \sqrt{x_4} + C_{11} + C_{21} x_1 + P_1 (x_1 - x_0) + M_1 x_2 \quad (3.1) \]

where \( C_{11}, C_{21}, P_1 \) and \( M_1 \) are real constants. We divided the coefficient of \( x_3 \) and the constant term into separate parts for practical purposes: \( C_{11} \) and \( C_{21} \) will depend on the other constants that define (1.1) while \( P_1 \) and \( M_1 \) can be freely chosen (as tuning parameters) and have an important part to play in stabilization.

Take now \( u_1 = \alpha_1 h_1 + \beta_1 L_f h_1 \) where \( L_f h_1 \) is the Lie derivative of \( h_1 \) along (1.1). Thus a tedious but not too difficult calculus gives

\[ u_1(x_1, x_2, x_3, x_4, x_5) = \alpha_1 (V_0 + Sx_1) \sqrt{p_1 - x_3} - \]

\[ (V_0 - Sx_1) \sqrt{x_4} + C_{11} + C_{21} x_1 + 
+ P_1 (x_1 - x_0) + M_1 x_2) + \beta_1 x_2 \times 
(S \sqrt{p_1 - x_3} + S \sqrt{x_4} + C_{21} S + P_1 - \frac{f_r M_1}{m}) \]

\[ = \frac{k M_1 x_1 + SM_2 x_3 - SM_2 x_4}{m} \]

\[ + \frac{SB x_2}{2 \sqrt{p_3 - x_3} - \frac{SB x_2}{2 \sqrt{x_3}}} \quad (3.2) \]

In order to have \( u_1(x) = 0 \) for \( x \) in (1.4) we take in (3.2)

\[ C_{11} = -V_0 \left( \sqrt{p_1 - x_3} - \sqrt{x_4} \right) \]

\[ C_{21} = -\sqrt{p_3 - x_3} - \sqrt{x_3} \quad (3.3) \]

The real constants \( \alpha_1, \beta_1, M_1 \) and \( P_1 \) will be chosen such that \( u_1 \) makes system (1.1) fulfil the conditions from Theorem 2.1 that lead to stability of equilibria.

Following the same reasoning to (1.2), one obtains

\[ u_2(x_1, x_2, x_3, x_4, x_5) = \alpha_2 (V_0 + Sx_1) \sqrt{x_4} - \]

\[ (V_0 - Sx_1) \sqrt{p_2 - x_4} + C_{12} + C_{22} x_1 + 
+ P_2 (x_1 - x_0) + M_2 x_2) + \beta_2 x_2 \times 
(S \sqrt{x_4} + S \sqrt{p_2 - x_4} + C_{22} S + P_2 - \frac{f_r M_2}{m}) \]

\[ = \frac{k M_2 x_1 + SM_2 x_3 - SM_2 x_4}{m} \]

\[ + \frac{SB x_2}{2 \sqrt{p_3 - x_3} - \frac{SB x_2}{2 \sqrt{x_3}}} \quad (3.4) \]

with

\[ C_{12} = V_0 \left( \sqrt{p_2 - x_4} - \sqrt{x_4} \right) \]

\[ C_{22} = -\sqrt{p_3 - x_3} - \sqrt{x_3} \quad (3.5) \]

Again, \( \alpha_2, \beta_2, M_2 \) and \( P_2 \) are real constants to be chosen such that the stability criterion in Theorem 2.1 be fulfilled, that is, the corresponding polynomials \( Q_1 \) be stable.

### IV. SIMULATION RESULTS

The choice of the key parameters \( (M, P, \alpha, \beta) \) in the synthesis of control laws \( u_1 \) and \( u_2 \) is simply conceived as a trial and error process of searching and finding a
configuration \((M, P, \alpha, \beta)\) to simultaneously satisfy the following conditions: 1) stability for the whole admissible domain \((p, x_0)\) of initial conditions; 2) unsaturated control \(u\), \(|u| \leq 10 \text{ V}\), in numerical simulations; 3) basin of attraction as large as possible, thus the conservation of stability to relatively large perturbations of equilibria (initial conditions of the solutions); and 4) the survival of the regulator problem solution when simulation is switched from continuous control law to a zero hold control (meaning constant control on a sampling time period).

The stability of the equilibrium point \((p, x_0)\) in the systems (1.1) and (1.2) is ensured by Theorem 2.1, if the polynomial \(Q_1\) in (2.5) is stable. Let \(Q_1(\lambda)\) be given by (see the developing of the characteristic equation for (2.4))

\[
Q_1(\lambda) = \lambda^4 + a_2(\lambda) \lambda^3 + a_3(\lambda) \lambda^2 + a_4(\lambda) \lambda + a_5(\lambda, x_0) \tag{4.1}
\]

From the Routh-Hurwitz criterion in its improvement given by Lienard-Chiepart (see [26]), \(Q_1\) is stable if and only if

\[
\begin{align*}
a_1 > 0, & \quad a_2(p, x_0) > 0, \quad a_3(p, x_0) > 0, \\
a_4(p, x_0) > 0, & \quad D_3 = a_1 a_2 a_3 - a_2^2 a_4 > 0 \tag{4.2}
\end{align*}
\]

The basic values of the system parameters are (see also [11]): \(m = 60 \text{ kg}, f = 3000 \text{ Ns/m}, k = 10^5 \text{ N/m}, S = 10^{-3} \text{ m}^2, V_0 = 3 \times 10^{-5} \text{ m}^3, p_c = 2 \times 10^7 \text{ N/m}^2, B = 6 \times 10^8 \text{ N/m}^2, c_d = 0.6, \rho = 850 \text{ kg/m}^3, \rho_v = 10^{-4} \text{ m/V}\) (meaning an equivalent valve port width \(w = 0.85 \text{ mm}\) and a maximal opening length of rectangular valve port \(x_{\text{max}} = 1 \text{ mm}\) at maximal valve input voltage \(u_{\text{max}} = 10 \text{ V}\) and \(\tau = \sqrt{\frac{1}{573}} \text{ s}\). These data characterize a hydraulic servo integrated in the aileron control chain of a Romanian jet fighter planned in the 1980’s.

The stability in the whole admissible plane of basic parameters \(EHS (p, x_0)\) is ensured, for instance, by the configuration \((M, P, \alpha, \beta) = (0, -1350, 0.000001, 0.0005)\) of key parameters, for both the choices \(x_5 > 0\) (Fig. 2a)) and \(x_5 < 0\). The borders defined by the lines (1.3) are present in the top right corner of Fig. 2a) and the bottom left corner of Fig. 2b); this figure corresponds to a poor configuration \((M, P, \alpha, \beta) = (0, -10, 0.001, 50)\), in other words, with the occurrence of instability domains in the plane \((p, x_0)\).

Given this first stage investigation of stability by means of such charts, the control laws obtained in Section 3 were tested, in a second stage, in numerical simulations. Worthy to mention, the two systems (1.1) and (1.2) represent in fact a decomposition of the movement of a “real”, switching type, \(EHS\) mathematical model [11].

\[
\begin{align*}
\dot{x}_1 &= x_2; \\
\dot{x}_2 &= (-k x_1 - f, x_2 + S x_3 - S x_4)/m \\
\dot{x}_3 &= B \left( C x_3 \text{sign}(p_3 (1 + \text{sgn}(x_3)) - 2 x_4) + p_3 (1 + \text{sgn}(x_3)) - 2 x_4 \right) \\
\dot{x}_4 &= B \left( C x_4 \text{sign}(p_3 (1 - \text{sgn}(x_3)) - 2 x_4) + 2 x_4 \right) \\
\dot{x}_5 &= -\frac{1}{\tau} x_5 + \frac{k}{\tau} u, \quad \tau := c_d w \sqrt{2/\rho}
\end{align*}
\]

In consequence, a switching control based on the concatenation of the laws (3.2), (3.4) was proposed

\[
\begin{align*}
u(x_1, x_2, x_3, x_4, x_5) &= \theta(x_3) \mu_1(x_1, x_2, x_3, x_4, x_5) \\
&\quad + \theta(-x_3) \mu_2(x_1, x_2, x_3, x_4, x_5) \tag{4.4}
\end{align*}
\]

\(\theta(x) = 1\) for \(x > 0\), \(\theta(x) = 0\) for \(x \leq 0\).
Figure 3. Stable evolutions of the state variables for the real system (4.3) (time in seconds, on abscissa); a), b), c) - various initial perturbed equilibria

Remind the expressions (1.4) for the equilibrium points. Representative plots of the variables evolutions are shown in Fig. 3a) (for the perturbed equilibrium \( x_1 = x_0 = 0.02 \text{ m}, x_2 = 0.1 \text{ m/s}, x_3 = 3 p_1, p \text{ [Pa]}, x_4 = 0.001 \text{ m}, \text{ and } p = 0.7 \)), and in Fig. 3b) (for the perturbed equilibrium \( x_1 = x_0 = 0 \text{ m}, x_2 = 0.1 \text{ m/s}, x_3 = 3 p_1, p \text{ [Pa]}, x_4 = 3 p_1, p \text{ [Pa]}, x_5 = 0.001 \text{ m}, \text{ and } p = 0.5 \)). A pseudocontinuous Runge-Kutta integration was performed; in all these cases, no significant difference, versus sampled zero-hold control (with sampling time 0.001 s), has been observed in simulations. In Fig. 3c) can be also seen the effective switching of the variable \( x_5 \) between positive and negative values, thus the effective working of the control (4.4) is validated.

Figure 4. Stable evolutions of the state variables for the split system (1.1) (time in seconds, on abscissa)

Figure 5. Instable evolutions of the state variables for the split system (1.1) (time in seconds, on abscissa)

The split systems (1.1), (1.2) were also simulated, in closed loop, with the controls (3.2) and respectively (3.4), choosing the parameters \( \beta \), \( \alpha \), \( \alpha \), and initial conditions (perturbed equilibria) such that the signs of \( x_5 \) be
preserved during the simulation time. In these circumstances, the results of the simulations when the two systems (1.1) and (1.2) are considered separately are qualitatively similar to those obtained with the switching system (4.4); compare Fig. 4, and Fig. 3a), carried out in the same conditions. Remark in all the graphs the stable evolution of the variable $x_1$ to a certain nonzero value, in concordance with Theorem 2.1. A Fig. 5 was added, confirming once again the instability chart given in Fig. 2b), (for the perturbed equilibrium $x_1 = x_0 = 0.01 \text{ m}$, $x_2 = 0.1 \text{ m/s}$, $x_3 = kx_0/S + p_j p [\text{ Pa}]$, $x_4 = 0.5 p_j p [\text{ Pa}]$, $x_5 = 0.001 \text{ m}$, and $p = 0.8$).

V. CONCLUDING REMARKS

The research presented in this paper combines classical Lyapunov-Malkin stability theory for critical cases with geometric control synthesis, as applied to an EHS five dimensional mathematical model. Even if peculiar, the problem of synthesis of a control law to ensure stability of EHS equilibria is meaningful: for a servo actuating airplane flight controls, stability with respect to all admissible equilibria means stability of airplane evolution for a flight with constant altitude (associated to $k = 0$ and $P = 0.5$) or a turn with constant curvature radius ($k > 0$ and $P \neq 0.5$). Four key analysis-synthesis parameters have been obtained and used with this purpose, with the remarkable result that the whole domain $(p,x_0)$ of equilibrium points can be stabilized. This fact is validated by stability charts and numerical simulations.

Finally, it is to mention that the problem of basin of attraction size for the EHS equilibria wasn’t in attention of the present paper, but follow to be considered in a future work, by using specific tools.

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