An Algorithm For Designing Feedback Stabilizers of Nonlinear Polynomial Systems

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Abstract—The aim of this paper is to present a symbolic computational algorithm that will allow us to deal with the feedback stabilization problem for continuous nonlinear polynomial systems. The overall approach is based on a methodology that checks the positivity of a given polynomial.

Keywords: Nonlinear Systems, Computational Algebraic Methods, Positivity, Sum of Squares, Feedback Stabilization.

I. INTRODUCTION

In this paper we examine the feedback stabilization problem for a large variety of continuous nonlinear systems of the form

$$\dot{x} = \Phi(x, u), \quad (\Sigma)$$

where the components of $\Phi$, $\Phi_i$, are multivariable polynomials. According to the theory, if the linearization of $(\Sigma)$ at the origin is asymptotically controllable (i.e., all the uncontrollable eigenvalues have negative real parts), then the nonlinear system is locally asymptotically stabilizable [5], [14]. This means that we can find a feedback-law that makes the closed-loop system asymptotically stable to the origin. The aim of this paper is to calculate those feedback-laws computationally by means of certain symbolic algorithms. Our methodology resembles that of the applied computational tools used to solve control problems [13]. Specifically, we consider a system. Concretely, we seek for nonlinear state-feedbacks of the form $u = a(x)$, where $a(x)$ consists of multivariable polynomials, so that a proper Lyapunov function becomes negative. To solve the problem we develop the following algorithms:

The Formal Algorithm. This algorithm allows us to write a polynomial $p$ as follows:

$$p = \sum_{\mu = 1}^{k} c_{\mu}(W_{i,\sigma,\phi})L_{1,\mu}^1 \cdot L_{2,\mu}^2 \cdots L_{n,\mu}^n + R_{W}(x_1) \quad (1)$$

with

$$L_{1,\mu} = x_1$$
$$L_{2,\mu} = W_{2,1,\mu}x_1 + x_2$$
$$\vdots$$
$$L_{n,\mu} = W_{n,1,\mu}x_1 + W_{n,2,\mu}x_2 + \cdots + x_n$$

where the exponents $j_{a,b}$ are specific positive whole numbers, the quantities $W_{i,\sigma,\phi}$ are undetermined parameters that can take certain values, $c_{\mu}(W_{i,\sigma,\phi})$ the coefficients depending on the parameters $W_{i,\sigma,\phi}$ and $R_{W}(x_1)$ a polynomial of the single variable $x_1$, called the remainder. Equation (1) is called the Formal-Linear-Like-Factorization of $p$ and appeared firstly in [15]. The essential tool of this methodology is a continuous reduction of $p$, by means of an Euclidean division. This makes our method similar to others [9], [11]. Yet in our methodology we only deal with a specific polynomial, in addition to the Gröbner basis that works with a polynomial ideal.

The POS-Algorithm. This algorithm checks the positivity of a multivariable polynomial $p$. This task can be achieved by giving to $W_{i,\sigma,\phi}$ values such that all the non-square terms to be eliminated and the square terms have positive coefficients. A lot of work has been done in this direction and some of the results can be found in the following [1],[2],[3],[4],[6],[7],[8],[10],[12], to mention but a few.

The Feedback-GAS-Algorithm. This algorithm accepts as input the polynomials $\Phi_i$ and a Lyapunov Function $L$, and calculates the feedback connection $u = a(x)$. To achieve this we assume that $a(x)$ consists of a multivariable polynomial with parametric coefficients (we denote them by $A_{i_1,i_2,\ldots,i_n}$). Our aim is to determine those values of the parameters $W_{i,\sigma,\phi}$ and $A_{i_1,i_2,\ldots,i_n}$ that make the Lyapunov function negative along the trajectories of the closed-loop system. This guarantees the stability of the origin. The main merits of our method are the following:

1) It constitutes a pragmatic computational method. Indeed, the feedback-laws are derived from symbolic computational algorithms. Appropriate software has been created for this purpose, and all the examples presented in the current paper have been studied with the aid of this software.

2) It provides us with a whole class of admissible controllers, as opposed to only a single one.

3) It works even when the linearization of the nonlinear system has an uncontrollable eigenvalue with zero real part, [14].

4) Since the Lyapunov function, upon construction, is negative for every point, the stability is global.

Throughout this paper $R$ and $Z^+$ will denote the sets of...
real numbers and positive integers correspondingly.

II. THE ALGEBRAIC BACKGROUND

Let $R$ be the set of real numbers and $x_1, x_2, \ldots, x_n$ be variables. An expression of the form

$$p = \sum_{\lambda} c_{\lambda} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

where $c_{\lambda} \in R$ and some of the exponents are $0$, is called a polynomial. The set of all real polynomials in $x_1, x_2, \ldots, x_n$ is denoted by $R[x_1, x_2, \ldots, x_n]$. An element $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is called a monomial and an element $c_{\lambda} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is called a term. Let $\phi_{n,\lambda} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $\phi_{m,\mu} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ be two monomials. We defined the lexicographical order among monomials [9], as follows: we say that $\phi_{n,\lambda}$ is ordered less than $\phi_{m,\mu}$, denoted by $\phi_{n,\lambda} < \phi_{m,\mu}$, if either $n < m$ or $n = m$ and $a_{n,\lambda} < a_{m,\mu}$. Let $p$ be a given polynomial, ordered lexicographically. The term that corresponds to the maximum monomial is called the maximum term denoted by $\text{maxterm}(p)$. Throughout the paper, a variable $W_{i,j,k}$ taking values in $R$ is called an undetermined parameter. The set of undetermined parameters is denoted by $\mathcal{W}$. Let $p \in R[x_1, x_2, \ldots, x_n]$ be a polynomial with $n$ variables and $\mathcal{W} = \{W_{i,j,k}\}$ a set of undetermined parameters, taking values in $R$. A Formal-Linear-Like-Factorization of $p$ is an expression of the form (1), where the coefficients $c_{\mu}(W_{i,j,\sigma})$ are polynomial functions of the parameters $W_{i,j,\sigma} \in \mathcal{W}$ and the remainder $R_{\lambda}(x_1)$ is a polynomial only of the single variable $x_1$, with coefficients depending on the parameters $W_{i,j,\sigma}$, too. Some of the exponents $j_{1,\mu}, j_{2,\mu}, \ldots, j_{n,\mu}$ can be zero. The Formal-Linear-Like-Factorization of $p$ is denoted by $\text{FormallF}(p)$.

Example 2.1: We have the polynomial $p = 5x_1 - 7x_1 x_2 + 11x_1 x_3$. The Formal-Linear-Like-Factorization of $p$ is $p = 5x_1(3,1,3,1,1) + 7x_1 x_2(1,2,1,1,1) + 11x_1 x_3(1,2,1,1,1,1)$. The following theorem deals with the uniqueness of the Formal-Linear-Like-Factorization.

Theorem 2.1: For a given polynomial $p \in R[x_1, x_2, \ldots, x_n]$, the Formal $\text{FormallF}(p)$ is unique, under the assumption that the parameters $\mathcal{W} = \{W_{i,j,\sigma}\}$ are considered as constants.

Proof: If $p$ has only $x_1$-terms, the proof of the theorem is trivial, with $c_{\mu} = 0$ and $R = p$. Let us suppose that $p$ has at least one term other than the $x_1$-terms. Let us further suppose that $\lambda x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is its maximum term. This term appears also in the product: $c_{h} x_1^{h_1} x_2^{h_2} \cdots x_n^{h_n}$. By equating their coefficients, we calculate the quantity $c_{h}$, uniquely, actually $c_{h} = \lambda$. Repeating the same procedure for the term with the next higher order, we find an expression for the "next" coefficient $c_{h-1}$. Since this expression is a function of $c_{h}$ and some of the parameters $W_{i,j,\sigma}$ are considered as constants, we conclude that $c_{h-1}$ is also defined uniquely. By induction, we finally get coefficients $c_{\mu, \beta} = 1, \ldots, k$, all of which are uniquely determined. The polynomial $R$ consists only of $x_1$-terms. These terms arise either from the polynomial $p$ or from the products $c_{\mu} x_{1,\mu} x_1^{j_{1,\mu}} x_2^{j_{2,\mu}} \cdots x_n^{j_{n,\mu}}$. The unique determination of the coefficients $c_{\mu}$ entails the uniqueness of $R$, and the theorem has been proved.

What is of interest is the issue of constructing the Formal-Linear-Like-Factorization of a given polynomial. This can be done through the algorithm we describe below.

THE FORMAL ALGORITHM

Input: A multivariable polynomial $p$, a set of undetermined parameters $\mathcal{W} = \{W_{i,j,\sigma}\}$, taking values in $R$.

Initial Conditions: $k = 0$

Step 1: We set $k = k + 1$.

Step 2: We find the maximum term of $p$, $\text{maxterm}(p) = c_{1}x_{1,1}^{j_{1,1}} \cdots x_{n,1}^{j_{n,1}}$. The coefficient $c_{1}$, in the first iteration, is a constant number. Then, it depends on the set of parameters $\mathcal{W}$.

Step 3: We form the linear polynomials: $L_{1,k} = x_1, L_{2,k} = W_{1,2,1} x_1 + x_2, \ldots, L_{n,k} = W_{n,1,1} x_1 + W_{n,2,1} x_2 + W_{n,3,1} x_3 + \cdots + x_n$.

Step 4: We make the subtraction: $R_{k} = p - c_{1}L_{1,k} + c_{2}L_{2,k} + \cdots + c_{n}L_{n,k}$.

Step 5: If $R_{k}$ depends only on the variable $x_1$ \textbf{THEN} set $R = R_{k}$ and go to the output ELSE put $p = R_{k}$ and go to step 1.

Output: The quantities $R_{k}, c_{\mu}, L_{i,j}, j_{i,\mu}, \mu = 1, \ldots, k, i = 1, \ldots, n$

The following theorem proves the finiteness and the efficiency of the algorithm.

Theorem 2.2: The Formal Algorithm terminates after a finite number of steps. If $R \neq 0$, $R_{k}, j_{1,\mu}, j_{i,\mu}, \mu = 1, \ldots, k, i = 1, \ldots, n$ are its outputs, then:

$$\text{FormallF}(p) = \sum_{\mu=1}^{k} c_{\mu} L_{1,\mu} x_1^{j_{1,\mu}} \cdots x_{n,\mu}^{j_{n,\mu}} + R$$

Proof: Let $p$ be a multivariable polynomial and $z = x_1^{j_{1,\mu}} x_2^{j_{2,\mu}} \cdots x_n^{j_{n,\mu}}$ its maximum term. We follow the Formal Algorithm step by step. When $k = 1$, step 2 will give $c_{1} = \gamma$ and $j_{1,1} = a_{1,1}, j_{n,1} = a_{n,1}$. Taking into consideration the above values and the construction of the linear polynomials $L_{1,1,1}, L_{2,1,1}, \ldots, L_{n,1,1}$, step 4 will produce a new polynomial $R_{1}$, which will not contain the term $z$. Obviously, the maximum term of $R_{1}$, which is called $z_1$, will be ordered lower than $z$, $z_1 < z$, with respect to the order raised earlier. By induction, we get for the maximum terms $z > z_2 > \cdots$. This nest and the construction of the order will finally eliminate all but $x_1$-terms. This fact guarantees the termination of the algorithm. Substituting now reversely, we have successively:

$$p = c_{1} L_{1,1} x_1^{a_{1,1}} \cdots x_{n,1}^{a_{n,1}} + R_{1}, R_{1} = c_{2} L_{2,2} x_1^{a_{2,2}} \cdots x_{n,2}^{a_{n,2}} + R_{2}, \cdots, R_{k-1} = c_{k} L_{n,k} x_1^{a_{n,k}} \cdots x_{n,n}^{a_{n,n}} + R,$$ Combined these we get:

$$p = \sum_{\mu=1}^{k} c_{\mu} L_{1,\mu} x_1^{j_{1,\mu}} \cdots x_{n,\mu}^{j_{n,\mu}} + R,$$ which is the Formal-Linear-Like-Factorization upon request.

Example 2.2: We have the previous polynomial $p = 5x_1 - 7x_1 x_2 + 11x_1 x_3$. We want to find a Formal-Linear-Like-Factorization of $p$. In order to clarify our ideas we shall follow the Formal-Algorithm in detail. First, $\text{maxterm}(p) = 11x_1 x_3$, here $c_{1} = 11$ and
The POS-Algorithm

Input: A multivariable polynomial \( p \), a set of undetermined parameters \( W = \{W_{i,\sigma,\varphi}\} \), taking values in \( \mathbb{R} \).

Initial Conditions: \( S = \{\} \), \( E = \{\} \).

Step 1: By means of the FORMAL ALGORITHM we get the quantities \( R \), \( c_{p} \), \( L_{i,\mu}, j_{i,\mu}, \mu = 1, \ldots, k, i = 1, \ldots, n \). We denote the coefficients of the remainder \( R \) by \( c_{p}, \mu = k + 1, \ldots, k + h \), the number of terms of \( R \).

Step 2: REPEAT FOR \( \mu = 1, \ldots, k + h \)

IF some of the exponents \( j_{i,\mu}, i = 1, \ldots, n \) are odd numbers THEN \( O = O \cup \{c_{p}\} \) ELSE \( E = E \cup \{c_{p}\} \)

NEXT \( \mu \)

Step 3: Find the values of the parameters \( W_{j,\sigma,\varphi} \) that eliminate the "odd" coefficients and make the "even" coefficients positive. In other words, we construct the set \( S = \{W_{j,\sigma,\varphi} = \lambda_{j,\sigma,\varphi} \in \mathbb{R} : c_{p} = 0 \text{ and } c_{p} \geq 0 \} \).

Output: The set \( S \).

The proof of the following theorem is straightforward.

Theorem 2.3: Let \( p \in \mathbb{R}[x_{1},x_{2},\ldots,x_{n}] \) be a given multivariable polynomial. If \( S \) is non-void, then \( p \) is positive definite, and \( \text{FORMAL}\operatorname{LF}[p] |_{p} \) is a family of "sum of squares" expressions of \( p \).

We would like to make the following remarks in connection with the above algorithm:

Remark 2.1: (i) We can modify the algorithm so that the whole procedure is executed "together" with the Formal Algorithm, as opposed to after it. This will allow for a quicker implementation of the method. (ii) Let us suppose that we have a Formal - Linear - Factorization of a given polynomial \( p \). The coefficients of the first terms with odd exponents contain a small number of parameters (usually one or two). This means that they can be eliminated easily for some particular values of the \( W \)-parameters. By substituting those values into the other terms we decrease the number of parameters, thus simplifying the whole computational procedure significantly.

Clearly, the method is not computationally complex and can be carried out normally. (iii) If the output of the POS-ALGORITHM is \( S = \{\} \), this does not mean that the polynomial \( p \) is not positive or that another sum of squares does not exist. Nevertheless, our approach can be extended by using nonlinear "factors"; for instance, \( L_{n,k} = W_{n,1,k}x_{1} + W_{n,2,k}x_{2} + \cdots + W_{n,n,1,k}x_{n} + W_{n,n,2,k}x_{n} + \cdots + W_{n,(n-1-n,1),k}x_{n-1} + x_{n}^{2} \). In this case we can obtain a further sum of squares that can successfully deal with cases in which the current method fails. This will be undertaken in a future study.

Example 2.4: Let us consider the polynomial \( p = x^{2} - 2xy + 6y^{2} - 4yz + 3z^{2} \). Its Formal-Linear-Factorization is: \( 3(z + xW_{3,1,1} + yW_{3,2,1})^{2} + (y + xW_{2,1,2})(-6W_{2,1,2} - 4) (z + xW_{3,1,2} + yW_{3,2,2}) + x(6W_{3,2,1}W_{2,1,2} + 4W_{2,1,2} - 6W_{3,1,1}) (z + xW_{3,1,1} + yW_{3,2,1}) + (y + xW_{2,1,2})^{2} - (3W_{2,1,2} + 6W_{2,2,2}W_{2,1,2} + 4W_{2,2,2}) + 6(y + xW_{2,1,2}) + \frac{6W_{2,1,2}^{2}}{1} - 6W_{3,1,1}W_{3,2,1} + 6W_{3,1,2}W_{3,2,1} + 6W_{2,1,2}W_{2,2,2}W_{2,1,2} - 12W_{2,1,2}W_{2,2,2} - 6W_{3,2,1}W_{3,3,2} - 6W_{3,1,2}W_{3,3,2} - 6W_{2,1,2}W_{3,3,2} - 6W_{3,3,1}W_{3,3,3} - 2) + R \). (We do not include the entire remainder because of its size). In order to eliminate the first non-square term to appear, we set \( W_{2,2,1} = -4 \). This transforms the factorization as follows: \( 3(-\frac{3}{2}) = z + xW_{3,1,1})^{2} - 6zW_{3,1,1}(z + xW_{3,1,2} + yW_{3,2,3}) + \frac{1}{2}(y + xW_{2,1,2})^{2} \).
\[-\frac{3}{2}x(y + x) + x^2 = 14W_{2,1,4} - 6W_{3,1,1} - 9W_{3,1,1}W_{3,2,3} + 3 + R'.\]

The values \(W_{3,1,1} = 0\) and \(W_{2,1,4} = -\frac{3}{11}\) eliminate the other non-square terms and the factorization becomes \(p = 3(-\frac{2}{5} + z)^2 + \frac{2}{11}(\frac{2}{11} + y)^2 + \frac{1}{11}z^2\). Thus, the set \(S\) upon request is \(S = \{W_{3,2,1} = -\frac{3}{5}, W_{3,1,1} = 0, W_{2,1,4} = -\frac{3}{11}\}
and all the other parameters \(W_{i,j,\varphi}\) are free. The above “sum of squares” expression of \(p\) guarantees that \(p\) is positive.

III. THE FEEDBACK ASYMPTOTIC STABILIZATION

We are now in a position to apply the entire concept that was raised earlier sections to the problem of feedback asymptotically stabilizing a nonlinear system at a given equilibrium point. This is a well-known topic that has been discussed extensively in the literature [5], [14]. In this paper we adopt a computational approach. Specifically, let us have the continuous nonlinear system: \(\dot{x} = \Phi(x, u)\), where \(x = (x_1, x_2, \ldots, x_n)^T\) is the state vector, \(u = (u_1, u_2, \ldots, u_m)^T\) the input vector and \(\Phi = \{\Phi_1(x, u), \ldots, \Phi_n(x, u)\}\), where \(\Phi_i(x, u), i = 1, \ldots, n\) are multivariable polynomials of \((x_1, \ldots, x_n)\) and \((u_1, \ldots, u_m)\) without free terms. Obviously \((x^0, u^0) = (0, 0)\) is an equilibrium point. Let us denote by \((A, B)\) the linearization pair of this nonlinear system around the origin, (i.e. \(A = \frac{\partial \Phi}{\partial x}(0, 0), B = \frac{\partial \Phi}{\partial u}(0, 0)\)). As is already known, [14], if the linear system \((A, B)\) is asymptotically controllable, then the corresponding nonlinear system is locally asymptotically stable at the origin. This means that we can find a matrix \(F\) such that the feedback-law constructs a system \(\dot{x} = \Phi(x, F(x))\) which is locally asymptotically stable at the point \((0, 0)\). The main concern of this paper is to calculate this quantity \(F\) computationally, but with the following alterations:

1) \(F\) does not need to be linear (a matrix) but may also be nonlinear (a polynomial function). Actually, the problem under examination is that of finding a state feedback of the form \(u = a(x)\), with \(a(x) = \{a_1(x), \ldots, a_m(x)\}\) and \(a_i(x), i = 1, \ldots, m\) multivariable polynomials too, where the corresponding closed-loop system \(\dot{x} = \Phi(x, a(x))\) has a global asymptotically stable equilibrium at \((x^0, u^0) = (0, 0)\).

2) Some of the uncontrollable eigenvalues of \((A, B)\) can have zero real parts.

3) The stability is not local but global.

At this point we introduce the algorithm in order to deal with the feedback asymptotic stabilization problem.

THE FEEDBACK-GAS ALGORITHM

Input: The polynomials \(\Phi(x, u)\), a polynomial Lyapunov function \(L\), the degree of the feedback law upon request.

Step 1: We define the feedback law \(u = a(x)\), \(a(x) = \{a_1(x), a_2(x), \ldots, a_n(x)\}\), with

\[a_j(x) = \sum_{i=1}^n A_{j,i}^{(j)} x_{i1} + \sum_{(i_1,i_2) = (1,1)} (n,n,...,n) A_{(i_1,i_2)}^{(j)} x_{i1} x_{i2} + \cdots + \sum_{(i_1,i_2,...,i_n) = (1,1)} A_{(i_1,i_2,...,i_n)}^{(j)} x_{i1} x_{i2} \cdots x_{i_n}\]

and \(A_{(i_1,i_2,...,i_n)}^{(j)}\) unknown parameters taking values in \(\mathbb{R}\).

Step 2: We define the quantity:

\(V = -\frac{\partial L}{\partial x}(\Phi(x, a(x))\), \(\Phi_2(x, \ldots, \Phi_n(x, a(x))\)

Step 3: By means of the POS-ALGORITHM we construct the set \(\mathcal{V}\), consisting of those values of the parameters for which \(V\) is positive.

Output: The set \(\mathcal{V}\).

Theorem 3.1: Let us have the nonlinear continuous system: \(\dot{x} = \Phi(x, u)\). Let \(\mathcal{V}\) be the output of the FEEDBACK-GAS-Algorithm. If \(\mathcal{V} \neq \emptyset\) then the set of feedback laws \(u = a(x)\) make the origin globally asymptotically stable.

Proof: The positive definiteness of the quantity \(V\) guarantees that the Lyapunov function \(L\) decreases along the trajectories of the closed-loop system \(\dot{x} = \Phi(x, a(x))\) and, therefore, the origin is asymptotically stable. Given that this is the case for every point, the origin is globally asymptotically stable.

Example 3.1: The angular momentum of a rigid body controlled by two independent torques can be described, following some simplification, through the equations: \(\dot{x} = \Phi(x, F(x))\), with \(x = [x_1, x_2, x_3, u = [u_1(t), u_2(t)]\)

\(\Phi_1(x_1, x_2, x_3) = a_1x_1x_3 + u_1\), \(\Phi_2(x_1, x_2, x_3) = a_2x_1x_3 + u_2\), \(\Phi_3(x_1, x_2, x_3) = a_3x_1x_2\). The quantities \(a_1, a_2, a_3\) are certain constants and \(a_3 \neq 0\), (14), page 176. It can be proved that this system can be globally stabilized about \(x = 0\) and \(u = 0\). A specific feedback can be constructed by following certain methods [14]. Following the steps of the Feedback-GAS-Algorithm we select a Lyapunov function of the form \(L = x_1^2 + x_2^2 + x_3^2\), and a pair of feedback-laws of the forms: \(u_1 = A_1x_1 + B_1x_2 + \Gamma_1x_3 + \Delta_1x_2x_3\), \(u_2 = A_2x_1 + B_2x_2 + \Gamma_2x_3 + \Delta_2x_1x_3\). Then, we define the quantity:

\(V = -\frac{\partial L}{\partial x}(\Phi_1(x_1, x_2, x_3)), \Phi_2(x_1, x_2, x_3) + \Phi_3(x_1, x_2, x_3), \Phi_4(x_1, x_2, x_3, \Delta_1x_2x_3)\)

\(= -\frac{\partial L}{\partial x}(\Phi_1(x_1, x_2, x_3)) - \Phi_2(x_1, x_2, x_3) + \Phi_3(x_1, x_2, x_3)\).

The Formal Algorithm will give the following Formal-Like-Factorization of \(V\):

\(x_1^2 + x_2^2 + x_3^2 = \frac{1}{2}((a_1 + a_2 + a_3 + \Delta_1 + \Delta_2)[W_2,1,1](x_1 + x_2)W_3,1,1 + x_2 W_2,2,3)\)

\(+ (x_2 + x_1 W_2,2,1)(W_3,1,1 + W_2,1,1 W_3,2,1 - 2W_2,1,5)\)

\(W_3,2,1 - W_1,1 W_3,2,3)\)

\(x_1^2 + x_2^2 + x_3^2 = \frac{1}{2}((a_1 + a_2 + a_3 + \Delta_1 + \Delta_2)[W_2,1,1](x_1 + x_2)W_3,1,1 + x_2 W_2,2,3)\)

\(+ (x_2 + x_1 W_2,2,1)(W_3,1,1 - x_1 W_2,1,1)\)

\(x_1^2 + x_2^2 + x_3^2 = \frac{1}{2}((a_1 + a_2 + a_3 + \Delta_1 + \Delta_2)[W_2,1,1](x_1 + x_2)W_3,1,1 + x_2 W_2,2,3) - (\Gamma_2 W_2,1,2 - \Gamma_1)(x_3 + x_1 W_3,1,1 + x_2 W_2,2,1)\)

\(x_1^2 + x_2^2 + x_3^2 = \frac{1}{2}((a_1 + a_2 + a_3 + \Delta_1 + \Delta_2)[W_2,1,1](x_1 + x_2)W_3,1,1 + x_2 W_2,2,3) - (\Gamma_2 W_2,1,2 - \Gamma_1)(x_3 + x_1 W_3,1,1 + x_2 W_2,2,1)\)
\[ \Gamma_2(x_2 + x_1 W_{2,1,2}) + x_3 W_{3,1,2} + x_2 W_{3,2,2} + R \]

where \( R \) is the remainder. The values of the parameters that eliminate the non-squares terms of the above expressions are \( W_{2,1,1} = 0, W_{3,2,1} = 0, W_{3,1,1} = 0, W_{2,1,4} = \frac{-A_2 + B_1}{2B_2}, \Gamma_1 = 0, \Gamma_2 = 0, \Delta_1 = -\Delta_2 - a_1 - a_2 - a_3 \) and \( A_1, A_2, B_1, B_2, \Delta_2, \) arbitrary.

For those specific values the formal LinearFactorization of \( \Gamma_2 \) becomes:
\[ \left( \frac{A_2^2 + 2B_2 A_2 + B_2^2 + 4A_2 B_2}{2B_2} \right)^2 \]

In order to get positive coefficients we further demand \( B_2 < 0 \) and \( A_1 < \frac{A_2^2 + 2B_2 A_2 + B_2^2}{2B_2} \). These conditions mean that the Lyapunov function decreases along the trajectories of the system, which guarantees the asymptotic stability of the origin. The family of feedback laws is given by the relations \( u_1 = A_1 x_1 + B_1 x_2 + \Delta_1 x_1 x_2, u_2 = A_2 x_1 + B_2 x_2 + \Delta_2 x_1 x_2 \) with \( B_2 < 0, A_1 < \frac{A_2^2 + 2B_2 A_2 + B_2^2}{2B_2}, \Delta_1 = \Delta_2 - a_1 - a_2 - a_3 \) and \( A_2, B_1, \Delta_2 \) arbitrary. We can also obtain other classes of feedback, by choosing other expressions for the parameters \( u_1, u_2 \) or other Lyapunov function.

Example 3.2: Let us suppose that we have the nonlinear system:
\[ \dot{x} = \Phi(x, u) = \begin{bmatrix} \Phi_1(x, u) \\ \Phi_2(x, u) \end{bmatrix} \]

with
\[ \Phi_1 = 4x + 8y - \frac{11}{10}x^2 + 3y u_1 - \frac{5}{2} u_1 u_2 - 4u_2 + 5x u_2, \]
\[ \Phi_2 = -\frac{7}{2} x^2 - 3y^2 - \frac{5}{2} u_2 x, \]
where \( x = [x(t), y(t)] \) is the state of the system, consisting of two functions and \( u = [u_1(t), u_2(t)] \) the input vector. We can easily check that the origin is an unstable equilibrium point for the system. Furthermore, the linearization of this system is not asymptotically controllable, since the polynomial \( \chi \) has a root with zero real part (see [14] for details). Following the steps of the Feedback-GAS Algorithm, we choose a pair of feedback laws of the form: \( u_1 = A_1 x + B_1 y + \Gamma_1 x y, u_2 = A_2 x + B_2 y + \Gamma_2 x y \). Then, we define the quantity
\[ V = -\Phi_1 (x, y, A_1 x + B_1 y + \Gamma_1 x y, A_2 x + B_2 y + \Gamma_2 x y) x - \Phi_2 (x, y, A_1 x + B_1 y + \Gamma_1 x y, A_2 x + B_2 y + \Gamma_2 x y) y \]

The Formal Algorithm will give the following Formal Linear-Factorization of \( V \):