System Modeling of MEMS Gyroscopes

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Abstract—This paper presents a systematic approach to obtain governing equations of Micro Electro Mechanical Gyroscopes. The focus will be on vibratory type and its mechanical parameters, involved in modeling and designing, will be discussed due to their deep impact on overall performance of Gyroscopes. Natural resonant frequency, fabrication imperfections and their source will also be introduced and taken into account. Finally, governing equations will be obtained and used to derive the dynamic equations of the system for driven and sense axes of MEMS Gyroscopes which are widely used as a basis for implementation of control schemes. Simulation results will be presented to show the compliance of the derived dynamic equations with a standard second order system.

Keywords: Modeling, MEMS, MEMS Vibratory Gyroscopes

I. INTRODUCTION

Most Micro Electro Mechanical (MEMS) Gyroscopes are vibratory rate gyroscopes, which have no rotating parts that require bearings, and hence they can be easily miniaturized and batch fabricated using micromachining techniques. These structures fabricated on polysilicon or crystal silicon, and their main mechanical component is a two degree of freedom vibrating proof mass, which is capable of oscillating on two directions in a plane. Their operation is based on the Coriolis effect. When the gyroscope is subjected to an angular velocity along an axis (input axis) orthogonal to the axis of initial oscillation (driven axis), the Coriolis effect transfers energy from one vibrating mode to another. The response of the second vibrating mode, which is along a third axis (sense axis) orthogonal to the previous two, provides information about the applied angular velocity [1].

Ideally, the vibrating modes of a MEMS gyroscope are supposed to remain mechanically uncoupled, their natural frequencies should be matched, and the gyroscope’s output should only be sensitive to angular velocity [2]. In practice, however, due to fabrication imperfections and variations in environmental parameters, a coupling between the two mechanical vibration modes through off-diagonal terms in the stiffness and damping matrices occurs which cause mismatch for driven and sense modes frequencies and thus results in degrading the gyroscope’s performance and therefore causes a false output.

In this paper, we study the configuration of MEMS vibratory Gyros, their mechanical design, and by introducing basics and mechanics involved in MEMS Gyros, we will derive the governing equations of these systems and model them based on a second order system.
the load $F_x$ is applied to the end of the guided-end beam in X-axis, the deflection, then, is given by (1), which corresponds to spring constant as stated in (2) in which $E$ is modulus of elasticity.

$$\delta_x = \frac{F_xL^3}{Ew^3h}$$  \hspace{1cm} (1)

$$k_x = \frac{Ew^3h}{L^3}$$  \hspace{1cm} (2)

To make the suspension stiff in the uncontrolled axis direction, a high aspect ratio between the controlled and uncontrolled axes is desirable. Since the spring constant ratio is approximately equal to the height-to-width aspect ratio squared, we can achieve this goal by manipulating the dimensions of the spring \[3,4\]. This is true in absence of residual axial stress. In the event that there is an axial stress along the fixed-guided beam, the stiffness of the beam increases. To calculate the stiffness of the tensioned beam, more complex beam analysis is required \[5\].

Normally, hairpin-type spring suspensions are used in a linear vibratory gyroscope since it is easy to set the X and Y-axes mode frequencies independently. Fig. 3 illustrates an example of hairpin-type suspension configuration.

The beam width is typically set to the minimum allowable width in the fabrication process. The separation $c$ between beams is set to the constant gap value of the process, to minimize footing and over-etching of the beams. Lengths $a$ and $d$ mainly affect the X-axis frequency while length $b$ affects the Y-axis frequency.

Equation (3) describes the X and Y-axes stiffness of suspension as a function of the lengths $a, b, c$ and $d$, in which, $a = d$ and $c \ll \langle a, b \rangle$ \[4\].

$$k_x = \frac{wh^3E}{4a^2(5a + 3b)}$$

$$k_y = \frac{wh^3E}{2b^3(4 - \frac{3b}{a + b})}$$  \hspace{1cm} (3)

B. Damping

Damping in micromachined structures can arise from viscoelastic strain in the mechanical element or from gas flowing around the moving structure. In most cases, gas damping dominates. Damping can also arise from structural energy dissipation and typically, the structural damping component is neglected since its magnitude is orders lower than that of the viscous damping component.

As the proof mass in MEMS gyroscopes moves in either the vertical or the horizontal planes, it experiences damping due to viscous effects of the air surrounding it. The two most frequently used models for damping are Couette flow damping and squeeze film damping.

Couette flow damping occurs when two plates slide parallel to each other. Such is the case of the damping between the proof mass and the substrate, and between the inter-digitated comb fingers used as electro static driving mechanism. We also have this situation in sensing mode, but there, squeeze film damping can better model the damping, which occurs between the comb fingers.

Assuming a Newtonian gas, the Couette flow damping coefficient is given by (4) where, $\mu$ is the gas viscosity (air in this case), $A$ is the plate area and $t$ is the film thickness \[3,4\].

$$d_c = \frac{\mu A}{t}$$  \hspace{1cm} (4)

Squeeze film damping occurs when two plates approach vertically each other. Such is the case of the damping between the proof mass and fixed electrodes. In addition, inter-digitated comb fingers will experience this damping as the distance between finger sidewalls’ changes. As the aspect ratio of the comb finger gap increases, this becomes the dominant source of damping in the sense mode of vibration.

A gas, which is trapped between two approaching plates, develops an average pressure, which is greater than the ambient pressure. Unlike Couette flow damping, squeeze film may exhibit both damping and stiffness effects depending on the compressibility of the film and the relative velocity of the two plates. This dual effect could happen depending on how much of the gas is compressed relative to how much of it is squeezed out between the plates. For fast moving, large plates, with very narrow separation, the gas is mostly compressed and behaves much like a spring. For slow moving, small and well separated plates, the gas is mostly squeezed out which results in a viscous damping effect. This ratio of compression to squeezing is captured in the squeeze number, $\sigma$, as described in (5).

$$\sigma = \frac{12\mu l^2\omega}{pt^2}$$  \hspace{1cm} (5)

Where $t$ is the thickness or gap, $l$ is the lateral dimension of the parallel plates, $\omega$ is the frequency of the plates oscillation, and $p$ is the ambient pressure.

For large $\sigma$, the gas is mostly compressed, for small $\sigma$, it is mostly squeezed. For thin film micromachined parallel plates used in the gyroscope, the squeeze number is small. This viscous damping could be
modeled using Reynolds gas-film equations [7] and Hagen-Poiseuille law [5].

Given the geometry of thin film gyroscopes, the squeeze film number will be small and the aspect ratio will be quite large \( \beta = \frac{w}{l} \). This allows the damping coefficient to be estimated as shown in (6) using Hagen-Poiseuille law and by assuming the shape factor for the typical surface micromachined gyroscopes in this method to be approximately 1/2 to 2/3 [3]:

\[
d_s = \frac{\gamma \nu A t}{t^3}
\]

(6)

C. Resonant frequencies

The placement of the sense and driven mode resonant frequencies is made in a way to achieve two objectives:

The driven mode resonant frequency should be higher than the bandwidth of other sources of acceleration to make Coriolis acceleration distinct from the others. Design of this resonant frequency in range of 10-20 KHz in addition to proper packaging of MEMS gyroscopes that limits the bandwidth of extraneous accelerations, can well do the separation.

The sense mode will have a tuning range determined by the voltages applied to the device. The design objective for the sense mode resonant frequency is to ensure that its tuning range includes the driven mode resonant frequency and that its frequency is somewhat higher than the driven mode resonant frequency because it will be lowered by electrostatic forces.

To accurately place resonant frequencies, there are different structures to be used; one of them is dual folded suspension resonance, which is illustrated in Fig. 4. The advantage of this design is that the residual stress will not be a factor in determining the driven mode resonant frequency and that its frequency is somewhat higher than the driven mode resonant frequency because it will be lowered by electrostatic forces.

To accurately place resonant frequencies, there are different structures to be used; one of them is dual folded suspension resonance, which is illustrated in Fig. 4. The advantage of this design is that the residual stress will not be a factor in determining the driven mode resonant frequency and makes the resonant frequency more a function of geometry defined by layout rather than that of process variables. Also in this approach, the outer platform is nominally stationary as the inner proof mass is oscillated. Measuring the displacement of a stationary platform is easier than measuring the displacement of a moving proof mass and yielding a better measurement of rotation rate.

Resonant frequencies of interest for this structure are shown in (7) and (8).

It is critical to have control over the compliance of the suspension. From the equations governing suspension stiffness, the most critical parameters to control are the ratio of beam width to beam length and residual stress. Fortunately, in gyroscope design, the relative placement of resonant frequencies is more important than the absolute value of the resonant frequencies [3].

D. Electrostatic spring rates

In order to drive the proof mass along its driven axis, electrostatic mechanisms are used. If the electrostatic force has a non-zero derivative with respect to displacement, the result is an effective spring rate. For the parallel plate capacitor system as electrostatic drive mechanism, as shown schematically in Fig. 5, the effective spring rate matrix can be found as described in (9) [3].

\[
k_{\text{plate}} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\varepsilon_0 z_0}{2(y_0 + y)^2} \\ \frac{\varepsilon_0 z_0}{2(y_0 + y)^3} & -\frac{\varepsilon_0 z_0(x_0 + x)}{(y_0 + y)^3} \end{bmatrix} V^2
\]

(9)

\( k_{xx} \) is zero implying that there is no electrostatic spring rate along the X-axis. The important element of this matrix is \( k_{yy} \). The sign of the Y-axis spring rate is opposite of the mechanical springs, thus the electrostatic attraction between parallel plates results in an electrostatic negative spring.

IV. DYNAMICS OF MEMS GYROSCOPES

First, we need to obtain accelerations experienced by a moving body in a rotating reference frame.

Given \( \hat{\mathbf{F}}_B \), the position vector of the moving body relative to a rotating reference frame \{B\} and the position vector of the rotating reference frame relative to an inertial reference frame \{e\} by orientation and position \( \hat{\mathbf{O}}, \hat{\mathbf{R}} \), the acceleration of the moving body can be calculated with respect to the inertial reference frame.

\[
\omega_x = \sqrt{\frac{2Et}{m_1 \left( w_1 \right)^3}}
\]

(7)

\[
\omega_y = \sqrt{\frac{2Et}{m_1 + m_2 \left( w_2 \right)^3}}
\]

(8)

Figure 5. The geometry of the parallel plate capacitance
Considering the time derivative of a vector measured in two reference frames as stated in (10) [8], we can now calculate the accelerations of an object moving in a rotating reference frame as in (11).
\[
\begin{align*}
\ddot{\mathbf{r}}_g(t) &= \ddot{\mathbf{r}}_b(t) + \dot{\mathbf{O}}(t) \times \dot{\mathbf{r}}_b(t) \\
\ddot{\mathbf{r}}_a &= \ddot{\mathbf{r}}_b + \dot{\mathbf{O}} \times \dot{\mathbf{r}}_b + \dot{\mathbf{O}} \times (\dot{\mathbf{O}} \times \dot{\mathbf{r}}_b)
\end{align*}
\]  

(10)

(11)

Assume that the Gyro frame \{g\} is rotated with respect to inertial frame \{e\} by the angular velocity vector \(\dot{\mathbf{O}}\), then the equation of motion for the proof mass of the gyroscope can be described by (12) using Newton’s second law of motion;
\[
\begin{align*}
\mathbf{F} &= m(\ddot{\mathbf{r}}_b + \dot{\mathbf{O}} \times \dot{\mathbf{r}}_b + \dot{\mathbf{O}} \times (\dot{\mathbf{O}} \times \dot{\mathbf{r}}_b))
\end{align*}
\]  

(12)

Where \(\mathbf{F}_0\) is the position vector of the origin of the Gyro frame with respect to the origin of the inertial frame, \(\mathbf{F}\) is the position vector of the proof mass with respect to the origin of the Gyro frame and \(\mathbf{F}\) is the total applied force to the proof mass which includes spring, damping and control forces.

The first term of (12) is the linear acceleration of the Gyro frame with respect to the inertial frame (or inertial force) and the second term is the linear acceleration of the mass in the Gyro frame. The third term is the Coriolis acceleration, which appears only if the equations of motion are written in the non-inertial frame. The fourth and fifth terms are the linear acceleration due to the angular acceleration and the centrifugal acceleration respectively [4].

The Coriolis term is the vectorial cross product of the input rotation rate and the velocity of the proof mass as measured in the rotating reference frame. Because the velocity is oriented along the X-axis and the sense mode is defined along the Y-axis, the gyro can only detect rotation rates about the Z-axis. Rotation rate \(\Omega\), which is our desired parameter to measure, can be inferred from measured Coriolis acceleration and the velocity of the moving body. This approach is the basis for all vibratory rate gyroscopes.

Each vector in (12) can be represented with respect to the Gyro frame \{g\} using its unit vectors along X, Y and Z-axes \{\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_y, \hat{\mathbf{u}}_z\}. The total force on Gyro consists of spring, damping and control forces hence we will have:
\[
\begin{align*}
\mathbf{F} &= \mathbf{f}_s + \mathbf{f}_d + \mathbf{f}_c
\end{align*}
\]  

(13)

\[
\begin{align*}
\mathbf{f}_s &= -k_1 \hat{\mathbf{u}}_x - k_2 \hat{\mathbf{u}}_y - k_3 \hat{\mathbf{u}}_z \\
\mathbf{f}_d &= -d_1 \hat{\mathbf{u}}_x - d_2 \hat{\mathbf{u}}_y - d_3 \hat{\mathbf{u}}_z \\
\mathbf{f}_c &= \tau_x \hat{\mathbf{u}}_x + \tau_y \hat{\mathbf{u}}_y + \tau_z \hat{\mathbf{u}}_z
\end{align*}
\]

Where \(x, y, z\) are the coordinates of the proof mass relative to the Gyro frame, \(d_{1,2,3}\) and \(k_{1,2,3}\) are damping and spring coefficients and \(\tau_{x,y,z}\) are control forces.

To measure the component of the angular velocity along the Z-axis, the motion of the proof mass can be constrained to be only in the X-Y plane by making the spring stiffness in the Z direction much larger than those of the X and Y directions. Thus, we arrive at (14-a) and (14-b).
\[
\begin{align*}
ma_x + m\ddot{x} + (k_1 - m(\Omega_1^2 + \Omega_2^2))x &+ m(\Omega_x \Omega_y - \Omega_z) = \tau_x + 2m\Omega_z \dot{y} \\
ma_y + m\ddot{y} + (k_2 - m(\Omega_1^2 + \Omega_2^2))y &+ m(\Omega_x \Omega_y + \Omega_z) = \tau_y - 2m\Omega_z \dot{x}
\end{align*}
\]  

(14-a)

(14-b)

Where \(\Omega_{x,y,z}\) are the angular velocity components along each axis of the Gyro frame. Assuming that the measured angular rate is almost constant over a long enough time interval and that linear accelerations are cancelled out, either as an offset from the output response or by applying counter-control forces, then the equation of motion of a gyroscope is simplified as follows [2].
\[
\begin{align*}
mx + d_1 x + (k_1 - m(\Omega_1^2 + \Omega_2^2))x &+ m\Omega_x \Omega_y = \tau_x + 2m\Omega_z \dot{y} \\
m\ddot{y} + d_2 \dot{y} + (k_2 - m(\Omega_1^2 + \Omega_2^2)) \dot{y} &+ m\Omega_x \Omega_y + \Omega_z = \tau_y - 2m\Omega_z \ddot{x}
\end{align*}
\]  

(15-a)

(15-b)

The last two terms in (15-a) and (15-b) are used to measure the angular rate \(\Omega_z\).

In an ideal gyroscope, only the component of angular rate along the Z-axis causes a dynamic coupling between the X and Y-axes, under the assumption \(\Omega_{x,y}^2 \approx \Omega_x \Omega_y \approx 0\). In practice, small fabrication imperfections occur and cause dynamic coupling between the X and Y-axes through the asymmetric spring and damping terms.

V. MODELING FABRICATION IMPERFECTIONS

Depending on the technology used to fabricate MEMS Gyros, different number of steps is involved in the process and, different fabrication tolerances can be achieved. These fabrication imperfections are reflected as asymmetry and anisoelectricity of the structure and cause coupling between X and Y-axes even under zero angular rate and also as misalignment of principal axes of elasticity from the intended axes of symmetry of the structure. Aside from the structural asymmetries, misalignment of the actuators with respect to the physical X and Y-axes and deviations between the position of the center of mass from the proof mass geometric center, contribute to the asymmetric spring terms [9]. Suppose that the principal spring constants are \(k_1\) and \(k_2\), and the principal spring axes are tilted by angle \(\varepsilon\) from the driven and sense axes because of the imperfections as shown in Fig. 6. Then, considering \(\{x', y'\}\) as the principal coordinates, we can obtain spring constants as stated in (16).
Figure 6. Modeling of spring constant and damping misalignment due to fabrication imperfections

$$
\begin{bmatrix}
F_x' \\
F_y' \\
F_x \\
F_y
\end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}
$$

(16)

Where,

$$
k_{xx} = k_1 \cos^2 \epsilon + k_2 \sin^2 \epsilon$$

$$
k_{yy} = (k_1 - k_2) \sin \epsilon \cos \epsilon$$

$$
k_{xy} = k_1 \sin^2 \epsilon + k_2 \cos^2 \epsilon$$

Similar to the stiffness case, asymmetric damping can arise, principally because of different quality factors or asymmetry in the aerodynamic drag, structural damping, etc., in two axes[1]. Coriolis off-set can also cause asymmetric damping terms. As in stiffness case, for damping we can write:

$$
d_{xx} = d_1 \cos^2 \epsilon + d_2 \sin^2 \epsilon$$

$$
d_{xy} = (d_1 - d_2) \sin \epsilon \cos \epsilon$$

$$
d_{yy} = d_1 \sin^2 \epsilon + d_2 \cos^2 \epsilon$$

(17)

VI. GOVERNING EQUATIONS

We can arrive at governing equations of MEMS gyroscopes by considering the impact of fabrication imperfections.

$$
\begin{align}
m\dddot{x} + d_{xx}\ddot{x} + d_{xy}\ddot{y} + (k_{xx} - m(\Omega_y^2 + \Omega_z^2))x \\
+ (k_{xy} + m\Omega_x \Omega_y)y &= \tau_x + 2m\Omega_z \dot{y} \\
m\dddot{y} + d_{xy}\ddot{x} + d_{yy}\ddot{y} + (k_{yy} - m(\Omega_x^2 + \Omega_z^2))y \\
+ (k_{yx} + m\Omega_y \Omega_x)x &= \tau_y - 2m\Omega_z \dot{x}
\end{align}
$$

(18-a) and (18-b) by the proof mass, will solve the problem. Since the angular rate is usually small as compared to the natural frequency of the system, and the proof mass is also small, the centrifugal force terms in (18-a) and (18-b) can be neglected or absorbed as part of the spring terms and as unknown variations. In the above case, (18-a) and (18-b) can be further simplified as [2,4]

$$
\begin{align}
m\dddot{x} + d_{xx}\ddot{x} + d_{xy}\ddot{y} + k_{xx}x + k_{yy}y &= \tau_x + 2m\Omega_z \dot{y} \\
m\dddot{y} + d_{xy}\ddot{x} + d_{yy}\ddot{y} + k_{yy}y + k_{xx}x &= \tau_y - 2m\Omega_z \dot{x}
\end{align}
$$

(19-a) and (19-b)

Non-dimensionalizing the equations of motion of a gyroscope is useful because the numerical simulation is easy and also produces a unified mathematical formulation for a large variety of gyroscope designs. Controllers can also be designed based on non-dimensional equations. Assuming $m$, $q_0$ and $\omega_0$, are the reference mass (proof mass), length and natural resonant frequency respectively, the desired equation will be as follows

$$
\begin{align}
\dddot{x} + \omega_0^2 x + \alpha_0^2 \dot{x} + \alpha_0^2 x &= \tau_x + 2\Omega_z \dot{y} \\
\dddot{y} + \omega_0^2 y + \alpha_0^2 \dot{y} + \alpha_0^2 y &= \tau_y - 2\Omega_z \dot{x}
\end{align}
$$

(20-a) and (20-b)

Where $Q_{x,y}$ are the X and Y-axes quality factors and:

$$
\omega_x = \sqrt{k_{xx} / m\omega_0^2} \quad , \quad \omega_y = \sqrt{k_{yy} / m\omega_0^2}
$$

$$
\omega_{xy} = k_{xy} / m\omega_0^2
$$

$$
d_{xx(new)} ← d_{xx(old) / m\omega_0^2} \quad , \quad \Omega_{x(new)} ← \Omega_{x(old) / \omega_0}
$$

$$
\tau_{x(new)} ← \tau_{x(old) / m\omega_0^2} \quad , \quad \tau_{y(new)} ← \tau_{y(old) / m\omega_0^2}
$$

Generally, sense axis is under force to balance control so the control signal can be used to measure the desired rotation rate, therefore, we can assume that $y = \dot{y} ≈ 0$ and simplifying (20-a) and (20-b) yielding system transfer function for the driven axis.

$$
G(s) = \frac{1}{m} \frac{1}{S^2 + 2\zeta\omega_n S + \omega_n^2}
$$

Where:

$$
\zeta = 1/2Q \quad \text{and} \quad \omega_n = \omega_c
$$

Using following values, the simulation results are shown in Fig 7 and 8:

$$
m = 10^{-6} Kg, \omega_c = 1.57 \times 10^3 \text{rad/ sec}, \zeta = 0.5 \times 10^{-3}
$$
VII. CONCLUSION

Equation (19) is the governing equation for a Z-axis MEMS gyroscope. Asymmetries in the stiffness and damping of a gyroscope result in an undesirable dynamic coupling between the two axes. Vibratory gyroscopes are designed to be extremely sensitive to the applied angular rate and therefore are equally sensitive to undesirable fabrication imperfections. In the governing equation, the spring coefficients $k_{xx}$ and $k_{yy}$, include the electrostatic spring softness previously discussed.

The simulation results for driven axis shows a lightly damped second-order system.

REFERENCES


