Output Feedback Control of an AUV with Experimental Results

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Abstract—This paper presents an output feedback controller for slender body underwater vehicles. The controller is derived using model based design techniques. Two separate control plant models are employed; a three DOF model accounting for the current loads acting on the vehicle and a five DOF model describing the vehicle dynamics. Furthermore, the transit model is based on the notion of constant propeller revolution resulting in a partly linearized model. The output feedback controller is proved to be uniformly globally asymptotically stable on the state space which it is defined. Experimental results from sea trials carried out with the Minesniper MkII demonstrate the performance of the proposed output feedback controller.

I. INTRODUCTION

A common challenge for most control design procedures for underwater vehicles is that the mathematical model often is highly coupled and nonlinear. However, dependent on the vehicle and the mission task, the model can be reduced and simplified to some extent. This can be e.g. to linearize the system about a constant forward speed, see e.g. [14] or reduce the number of states according to the properties of the vehicle, see e.g. [2] and [5]. Moreover, dependent on the application, nonlinearities can be removed from the model reducing the complexity of the system further. Unfortunately, poorly formulated control plant models (CPM)s that do not capture the important characteristics of the dynamic system, may cause reduced performance and also stability problems. Hence, when deriving the CPM, emphasize should be placed on stability and robustness issues related to the system in addition to simplifying the model such that analysis is feasible. A CPM is, according to [18], defined as a model that captures main characteristics of the physical system.

The main contribution of this paper is a uniformly globally asymptotically stable output feedback controller for slender body underwater vehicles with only position and orientation measurements. The derivation of the mathematical model is based on the approach presented in [16], proposing two co-working models; a 6 DOF vehicle model and a 3 DOF current model. Furthermore, an important objective of this work is to develop an observer-controller system that is easily implementable. Therefore, constant thruster revolution is applied, resulting in a semi-linearized model. However, due to the nonlinear coupling between the current model and the vehicle model, the stability analysis becomes quite involved, but the resulting observers and controller are quite simple and easily implementable.

This paper is organized as follows: A description of the mathematical modelling is shown in section II. The observer and controller design and analysis are given in Section III and IV, respectively. Furthermore, a case study on the Minesniper MkII with full scale experimental results is presented in Section V. Finally, conclusions are given in Section VI.

II. MATHEMATICAL MODELLING

Inspired by the modelling approach presented in [10], we decouple the surge motion from the rest of the model. This gives the following equation for surge

\[ m_{11} \ddot{u} + d_{11}(|u_r|)u_r = \tau_u(u_r) \]  

where \( m_{11} \) denotes the mass (included added mass), and \( d_{11}(|u_r|) \) represents the linear and nonlinear hydrodynamic damping. The relative surge velocity is given by \( u_r = u - u_c \), where \( u \) and \( u_c \) denote the body-fixed vehicle and current velocity, respectively. Furthermore, the force generated by the thrusters is given by [7]

\[ \tau_u(u_r) = -\beta_1 |u_r|n + \beta_2 |n|n \]

where \( n \) [rad/s] denotes the propeller revolution, and \( \beta_1, \beta_2 \) are positive constants. In transit, it is reasonable to determine a constant revolution which the forward thruster is operating on. This is mainly because obtaining a certain speed is of less importance compared to tracking of the orientation and depth. Therefore, employing a constant propeller revolution, it follows from (1) and (2) that the relative forward velocity \( u_r \) is constant in steady state, i.e. \( u_r(t) = U_r \). Since \( U_r = u(t) - u_c(t) \) this means that the even though \( u_c(t) \) is time-varying, it follows from (1) and (2) that \( u(t) \) will vary accordingly such that the relative velocity remains constant. We will in this paper consider AUVs in transit where the forward velocity is larger than the current velocity such that \( U_r > 0 \).

The CPM presented in this section is based on the approach introduced in [16]. Moreover, key properties of a slender body AUV are taken into account when deriving the model; port-starboard symmetry, self-stabilizing roll, and the fact that the length is much larger than the width.

CPM 1: By neglecting roll and applying a constant propeller set-point, the following CPM is proposed

\[ \dot{\theta} = J(\Theta)\nu \]  

\[ M \dot{\nu} + D(|\nu_r|)\nu_r + U_c C_1 \nu_r + u_c C_2 \nu_r \]  

\[ + g(\Theta) = \tau + J^{-1}(\Theta)\dot{b} \]  

\[ \dot{b} = -\kappa \lambda T^{-1}b \]
where \( \eta = [x, y, z, \theta, \psi]^T \) denotes the North-East-Down (NED)-frame position and orientation described by the pitch and yaw angles \( \Theta = [\theta, \psi]^T \). The matrix \( J(\Theta) \) transforms body-fixed vectors into NED coordinate frame, and \( g(\Theta) \) is the vector containing the gravity and buoyancy. For more details regarding \( J(\Theta) \) and \( g(\Theta) \), see e.g. [6]. The velocity vector \( \nu = [u, v, w, q, r]^T \) denotes the surge, sway, heave, pitch and yaw velocity, respectively. In the NED-frame, the current is described by the vector \( \nu_c = [u_c, v_c, w_c, 0_{1\times 2}]^T \). This vector is represented in the body-frame as \( \nu_c = J^{-1}(\Theta)\nu_c^b \), where \( \nu_c^b = [u_c, v_c, w_c, 0_{1\times 2}]^T \). Here, \( u_c \), \( v_c \) and \( w_c \) represent the current velocity in surge, sway and heave, respectively. This gives the following expression for the relative velocity \( \nu_r = \nu - \nu_c = [U_r, v_r, w_r, 0_{1\times 2}]^T \). In this paper we consider vehicles with control actuators in surge, pitch and yaw resulting in the following control vector \( \tau = [\tau_u, 0_{1\times 2}, \tau_\theta, \tau_\psi]^T \), which is common for underwater vehicles in transit. The total damping is given by a linear (\( D_l \)) and a nonlinear (\( D_{nl} \)) part according to \( D(\nu_r) = D_l + D_{nl}(\nu_r) \), where the nonlinear damping matrix is assumed diagonal. Including the dominant parts of the destabilizing Coriolis and centripetal forces and moments, the Coriolis elements yield

\[
\begin{align*}
C_{ij}^l &= C_{ij}^l = 0 \quad \text{where } i, j = 1, 5, \text{except} \\
C_{16}^l &= m - X_u, C_{35}^l = X_u - m, C_{53}^l = Z_w - X_\nu \\
C_{12}^l &= X_u - Y_v, C_{26}^l = C_{25}^l = m
\end{align*}
\]

Here, \( m \) denotes the vehicle’s rigid body mass, and \( X_u, Y_v \) etc. are the added mass coefficients. The bias \( b \) (3c) is included to compensate for unmodelled vehicle and thruster dynamics. Thus, the unmodelled coupling terms between surge and the remaining states, are assumed to be captured by the bias. It is modelled as a Markov process where \( k \) and \( \lambda \) are suitable positive constants, and \( T \) is a diagonal matrix containing the positive time constants.

**CPM 2:** This is a vessel model that captures the slowly varying effects caused by the current loads. The key task of this model is to function as a basis for observer design to obtain an estimate of the current velocity \( \nu_c \).

\[
\dot{\eta}_2 = R(\Theta)\nu_2 \\
M_2\nu_2 + D_2(\nu_2)\nu_2 = \tau_2 + R^T(\Theta)b_2 \\
b_2 = -\kappa T_{2}^-b_2
\]

where \( \nu_2 = [u_2, v_2, w_2]^T \) represents the vehicle velocities in surge, sway and heave, respectively, and \( \eta_2 = [x_2, y_2, z_2]^T \) denotes the vehicle position in the NED-frame. The matrices \( R(\Theta), M_2, D_2, T_2 \in \mathbb{R}^{3\times 3} \) are the top left sub-matrices of \( J(\Theta), M, D \) and \( T \) in (3), respectively. The control vector yields \( \tau_2 = [\tau_u, 0, 0]^T \). Underwater vehicles with no control actuators in sway and heave are incapable of counteracting the forces induced by the current in these directions. Hence, in CPM 2, it is clear that provided \( \tau_2 \), and since slender body gives that \( M_2 \) and \( D_2 \) are diagonal matrices, any non-zero velocities in sway and heave \((v_2, w_2)\) must originate from the bias \( b_2 \), which captures the slowly varying current forces and unmodelled dynamics. Furthermore, based on the surge model (1), we can determine the relative forward speed of the vehicle \( U_r \) given on the propeller revolution \( n \). Hence, by comparing \( U_r \) with \( u_2 \), the induced current velocities may be obtained according to the following

\[
u_c = u_2 - U_r, \quad v_c = v_2, \quad w_c = w_2
\]

**III. OBSERVER DESIGN**

In the following section we propose two separate nonlinear Luenberger observers for the CPMs presented in Section II.

**A. Preliminaries**

The following assumption and properties yield throughout the paper:

**A. 1** The pitch angle is limited by \( |\theta| < \frac{\pi}{2} \). For most underwater vehicles this is realistic given the inherent restoring moments preventing the vehicle from large pitch angles.

**P. 1** Only the position and the Euler angles, i.e. the vector \( \eta \), is measured.

**P. 2** The magnitude and direction of the current is unknown but upper bounded, i.e. there exists a constant \( \nu_c \in \mathbb{R}^3 \) such that \( \nu_c = \sup_{\nu_c} \parallel \nu_c(t) \parallel \). Therefore, since CPM 2 models the influence of the current, it follows that \( b_2 \) is bounded, since a non-zero \( b_2 \) only can be due to the current influence on the vehicle. Hence, \( \exists B_2 \in \mathbb{R}^+ \) such that \( b_2 = \sup_{\nu_c} \parallel b_2(t) \parallel \), □

We use the following notation in this paper. For any matrix \( A = A^T > 0 \), \( A_m \) and \( A_M \) represent the minimum and maximum eigenvalue of \( A \), respectively. Furthermore, \( \parallel \cdot \parallel \) denotes the Euclidian norm of a vector or a matrix.

In order to fully utilize the properties of the hydrodynamic damping, the 5 DOF damping matrix in CPM 1 is separated according to \( D(\nu_r) = D_d(\nu_r) + D_l \), where \( D_d(\nu_r) \equiv D_{nl}(\nu_r) + D_{nl}^d(t) \) is diagonal. Here, \( D_{nl}^d \) and \( D_l^d \) denote the diagonal and off-diagonal part of the linear damping matrix, respectively. The reason for this partition is that the linear damping matrix does not necessarily satisfy \( D_l + D_l^T > 0 \) for slender body AUVs. However, the following property yields:

**P. 3** \( D_d(s)|s| > 0, \forall s \in \mathbb{R}^5 \). Moreover, there exists a constant \( \delta_\nu \in \mathbb{R}^+ \) such that \( \parallel D_d(s)|s| \leq \delta_\nu \parallel s \parallel, \forall s \in \mathbb{R}^5 \). To simplify the notation in the following of this paper we define the function \( d(s) \equiv D_d(s)|s| \), which by employing the mean value theorem gives

\[
D_d(|b|)|b - D_d(|a|)|a = \frac{\partial}{\partial \epsilon} d(e_{b-a})\Big|_{\epsilon=b-a=e_0} = (b-a) \\
\leq \delta(|b-a|)
\]

where \( b, a, e, e_0 \in \mathbb{R}^5 \) and \( e_0 \) is on the line segment joining \( b \) and \( a \). Furthermore, due to linear hydrodynamic damping, there exists a constant \( \delta_m \in \mathbb{R}^+ \) such that \( 0 < \delta_m < \parallel \delta(\cdot) \parallel \).

**P. 4** The transformation matrix can be expressed as \( J(\Theta) = \text{diag}[R(\Theta), T_{\Theta}(\Theta)] \in \mathbb{R}^{5\times 5} \), where \( R(\Theta) \) satisfies \( R^T(\Theta) = R^{-1}(\Theta) \). Furthermore, under A.1, there exist constants \( \ell_1, \ell_2 \in \mathbb{R}^+ \) such that \( \forall s \in \mathbb{R}^2 \) yields

\[
\parallel T_{\Theta}(s) \parallel \leq \ell_1 \parallel s \parallel, \quad \parallel T_{\Theta}^{-1}(s) \parallel \leq \ell_2 \parallel s \parallel.
\]

For more details on \( J(\Theta) \), see [6, Ch. 2].
B. Current observer

The following Luenberger observer is proposed, where the gray box indicates that this is implemented in the control system.

\[
\dot{\hat{\nu}_2} = R(\Theta)\hat{\nu}_2 + \lambda L_2 \hat{\eta}_2 \\
M_2 \dot{\hat{\nu}_2} + D_2(\hat{\nu}_2)\hat{\nu}_2 = \tau_2 + R^T(\Theta)\hat{b}_2 + \lambda^2 M_2 R^T(\Theta)K_2 \hat{\eta}_2 \\
\hat{b}_2 = -\kappa \lambda T_2^{-1} \hat{b}_2 + \kappa \lambda^2 L_2 \hat{\eta}_2
\]

where \( \hat{\eta}_2 \equiv \eta_2 - \hat{\eta}_2 \), and \( L_2, K_2, K_{b2} \in \mathbb{R}^{3 \times 3} \) are positive definite and diagonal matrices. According to (5), the estimated current velocity is derived as \( \hat{\nu}_c = [\hat{\nu}_2 - U_r, \hat{\nu}_2, \hat{u}_2, 0, 0]^T \). Furthermore, we have that \( \hat{\nu}_2 \equiv \nu_2 - \hat{\nu}_2 \) and \( \hat{b}_2 \equiv b_2 - \hat{b}_2 \). Subtracting (6) from (4) an letting \( x_{C1} \equiv \frac{1}{\kappa} \hat{\eta}_2 \), \( x_{C2} \equiv \frac{1}{\kappa \lambda} \hat{b}_2 \) and \( x_{C3} \equiv \frac{1}{\kappa^2 \lambda^2} \hat{b}_2 \), the observer error dynamics become

\[
\dot{x}_{C1} = \lambda x_{C2} - \lambda L_2 x_{C1} \\
\dot{x}_{C2} = -M_2^{-1} \delta_2^\prime(x_{C2}) x_{C2} - \lambda K_2 x_{C1} + \kappa M_2^{-1} x_{C3} \\
\dot{x}_{C3} = -\kappa \lambda T_2^{-1} x_{C2} - \lambda \kappa K_{b2} x_{C1}
\]

where \( \delta_2^\prime(\cdot) \) refers to \( \delta \) in P.3. Notice that the right hand side of the system equations (7) includes the time-varying vector \( \Theta(t) \), while we only consider the \( [x_{C1}, x_{C2}, x_{C3}]^T \) dynamics. To circumvent this problem, we can consider \( \Theta(t) \) as a general time-varying signal using forward completeness as in [13]. The complete closed loop system can be proven forward complete. However, due to lack of space, this proof is not included in this paper. See e.g. [16] for details. Hence, we continue assuming that the time-varying vector \( \Theta(t) \) exists for all \( t \geq t_0 \). Let \( x_C \equiv [x_{C1}^T, x_{C2}^T, x_{C3}^T]^T \).

Proposition 1 The origin \( x_C = 0 \) of the observer error dynamics (7) is uniformly globally exponentially stable (UGES).

Proof : Following the same lines as in [9], it can be shown, by letting \( \kappa \) be sufficiently small, that \( x_C = 0 \) is USGES if \( \boxed{\Theta(t) \rightarrow \Theta} \).

Given this result, we have that the error variable \( \hat{x}_C(t) \) is bounded by

\[
\|x_C(t)\| \leq \left( k_2 / k_1 \right) \|x_C(t_0)\| e^{-(k_3 / 2k_2)(t-t_0)}
\]

where \( k_1, k_2 \) and \( k_3 \) are positive constants. Then, using P.2, we can obtain an upper bound on the current error velocity \( \hat{\nu}_C = \nu_C - \hat{\nu}_C \). Let \( \hat{\eta}_2(t_0) = \eta_2(t_0), \hat{\nu}_2(t_0) = 0, \hat{b}_2(t_0) = 0 \) and \( W_e \in \mathbb{R}_+ \). Then using (8), there exists a constant \( W_e \in \mathbb{R}_+ \) such that

\[
\|\hat{\nu}_C(t)\| \leq \left( k_2 / k_1 \right)^{\frac{1}{2}} (W_e + B_2) \| W_e \) \forall \ t \geq t_0
\]

C. The vehicle observer

In this section we derive a nonlinear Luenberger observer for the vehicle dynamics. To avoid technicalities using two frames (NED and body frame), CPM 1 is rewritten in NED-frame coordinates as follows

\[
\nu^e \triangleq \hat{\eta} = J(\Theta)\nu \\
M^* \hat{\nu}^e + D^*([\nu_2, |\nu_2|]) \hat{\nu}^e + U_r C_1^* \nu^e \\
+ u_c C_2^* \nu^e + g^* = J^T(\Theta) \tau + b \\
\hat{b} = -\kappa \lambda T^{-1} b - \kappa \lambda^2 K_\eta \hat{b}
\]

where \( \nu^e = [\dot{x}, \dot{y}, \dot{z}, \dot{\theta}, \dot{\psi}]^T \) denotes the NED-frame velocities. See e.g. [6, Ch. 3.3] for details regarding the matrix transformations.

P. 5 There exist sufficiently large constants \( c_i^*, c_i > 0 \) such that the Coriolis matrices are upper bounded according to

\[
\|C_i^*(x)\| \leq c_i^* \|x\|, \quad \|C_i(x)\| \leq c_i \|x\|, \quad i = 1, 2
\]

Inspired by the observer presented in [4], we propose the following observer

\[
\hat{\eta} = \hat{\nu}^e + \lambda \hat{L}\hat{\eta} \\
M \hat{\nu}^e + D^*([\hat{\nu}_2, |\hat{\nu}_2|]) \hat{\nu}^e + U_r C_1^* \hat{\nu}^e + \hat{u}_c C_2^* \hat{\nu}^e \\
+ g^* = J^T(\Theta) \tau + b + \lambda^2 M^* K^\eta \hat{b} \\
\hat{b} = -\kappa \lambda T^{-1} b - \kappa \lambda^2 K_\eta \hat{b}
\]

where \( \hat{\eta} \equiv \eta - \hat{\eta} \), and \( L, K, K_\eta \in \mathbb{R}^{5 \times 5} \) are positive definite and diagonal matrices. Subtracting (11) from (10) and using P.3 gives the following error dynamics in

\[
\hat{\eta} = \dot{\nu}^e - \lambda \hat{L}\hat{\eta} \\
M^* \hat{\nu}^e + \delta_4^\prime(\cdot) \hat{\nu}^e + (D_1^* + U_r C_1^* + u_c C_2^*) \hat{\nu}^e \\
= \hat{b} - \lambda^2 M^* K^\eta \hat{b} - \hat{u}_c C_2^* \hat{\nu}^e \\
\hat{b} = -\kappa \lambda T^{-1} b - \kappa \lambda^2 K_\eta \hat{b}
\]

where \( \hat{\nu} \equiv \nu - \hat{\nu} \) and \( \hat{b} \equiv b - \hat{b} \). Applying cascaded systems theory [15], we define the following perturbation vector

\[
g_p^* = [0_{5 \times 1}, -\hat{u}_c [C_2^* \nu^e]^T, 0_T^T, 0_{5 \times 1}]^T
\]

since \( g_p^* \) is proportional to the current estimation error. The nominal vehicle error observer error dynamics then become

\[
\hat{\eta} = \dot{\nu}^e - \lambda \hat{L}\hat{\eta} \\
M^* \hat{\nu}^e + \delta_4^\prime(\cdot) \hat{\nu}^e + (D_1^* + U_r C_1^* + u_c C_2^*) \hat{\nu}^e \\
= \hat{b} - \lambda^2 M^* K^\eta \hat{b} - \hat{u}_c C_2^* \hat{\nu}^e \\
\hat{b} = -\kappa \lambda T^{-1} b - \kappa \lambda^2 K_\eta \hat{b}
\]

Notice, however, that \( g_p^* \) also evolves linearly with the estimated velocity \( \hat{\nu}^e \), which growth is unknown. Hence, we apply the following assumption:

A. 2 There exists a constant \( V_r \in \mathbb{R}_+ \) such that the relative velocity is bounded according to \( V_r = \sup \|\nu^e(t)\| \).

This assumption will be lifted when the overall closed loop system, including the controller error dynamics, are analyzed. Under A.2, we have that \( \|g_p^*\| \leq |\hat{u}_c| c_2^*[V_r + \|\hat{\nu}^e\|] \), which only consists of error variables.

In order to fully utilize the dissipative property of the hydrodynamic damping, we will include the current estimation error in the upcoming stability analysis of the vehicle dynamics
observer. Let the nominal system error vector be defined as
\[ x_1 = \frac{\partial}{\partial \tilde{X}} \tilde{x}, \; x_2 = \frac{\partial}{\partial \tilde{D}} \tilde{y} \text{ and } x_3 = \frac{\partial}{\partial \tilde{C}} \tilde{v} \text{, where } \beta_c > 0 \] is a constant which is introduced to ensure stability in the following Lyapunov analysis. The nominal error dynamics, omitting the perturbation vector becomes
\[ \dot{x}_N = \lambda A x_N - M^{* -1} D^{* -1} x_N \]

\[ \approx \left[ \begin{array}{c} 0_d \cr f_{x_1}^{C_1} x_N \cr f_{x_2}^{C_2} x_N \end{array} \right] \]

where
\[ x_N \approx \left[ \begin{array}{c} x_C^T, x_o^T \end{array} \right]^T \]

\[ A = \left[ \begin{array}{ccc} -L & I & 0 \\ -K_2 & 0 & 0 \\ 0_3 & 0_3 & -K \end{array} \right] \]

The nonlinear and diagonal damping matrices are collected in \( D^{* T} \) using P.3 as follows
\[ D^{* T} = \left[ \begin{array}{c} 0_3 \cr 0_3 \cr \beta \delta_1(x) \cr \delta_2(x) \cr 0_5 \end{array} \right] \]

\[ M^{* T} = \text{diag}[0_3, 0_3, 0_3, 0_3, 0_5, 0_5] \]

\[ h = \left[ \begin{array}{c} 0_1 \\ 0_1 \\ 0_1 \\ 0_1 \end{array} \right] \]

Moreover, we have that
\[ f_{x_1}^{C_1} x_N = \kappa M^{-1} x_{C_1} \]
\[ f_{x_2}^{C_2} x_N = \lambda x_N + U_r^c + u_c C^2 \]

Notice that system (15) is autonomous since all the error variables, including the Euler angles \( \Theta \), are concatenated in the error vector \( x_N \). Consequently, we apply autonomous Lyapunov stability theory. The matrix \( A \) is Hurwitz and hence, there exist positive definite and symmetric matrices \( S \) and \( Q \) such that \( SA + A^T S = -Q \).

**Proposition 2** The origin \( x_o = [x_1^T, x_2^T, x_3^T] = 0 \) of the nominal observer error dynamics (14) is globally exponentially stable (GES) under A.2, and if the following condition is satisfied
\[ \lambda > W_o / Q_m \]

where \( W_o \triangleq \frac{2}{M} [(\delta^*_o + U_r^c + V_c^2)](1 + \beta_c) + 2\kappa \].

**Proof:** Consider the following positive definite, radially unbounded Lyapunov function candidate \( V_N(x_N) = x_N^T S x_N \). Differentiating \( V_N(x_N) \) with respect to time gives
\[ \dot{V}_N(x_N) = -\lambda x_N^T S x_N - 2 x_N^T M^{* -1} D^{* -1} x_N + 2 x_N^T \Phi(x_N) \]

Using (9) and P.5, \( \Phi(x_N) \) can be upper bounded as follows
\[ ||\Phi(x_N)|| \leq \frac{2}{M} \left[ ||\delta^*_o + U_r^c + V_c^2|| x_2 || + \beta_c || x_C^2 || \right] + 2\kappa || x_3 || \leq W_o || x_N || \]

Furthermore, under A.2 and letting \( \beta_c \) be sufficiently small, we have that the term containing the nonlinear damping satisfies
\[ x_N^T M^{* -1} D^{* -1} x_N \geq 0 \] for all \( x_N \in \mathbb{R}^{24} \), since \( M^{* -1} = M^{* T} > 0 \). We thus arrive at the following upper bound on the Lyapunov function derivative:
\[ V_N(x_N) \leq -||x_N||^2 (\lambda Q_m - W_o) \]. Using Lyapunov theory [11], it follows that if (16) is satisfied, the origin \( x_N = 0 \) of the nominal observer error dynamics (15) is GES. Furthermore, since \( x_C = 0 \) is proven UGES in Section III-B and \( ||x_o|| \leq ||x_N|| \), it follows that \( ||x_o(t)|| \leq k_{x_o} ||x_o(t_0)|| e^{-\lambda_{x_o} t} \), where \( k_{x_o} \) and \( \lambda_{x_o} \) are sufficiently large and small positive constants, respectively. Hence, \( x_o = 0 \) of the nominal observer error dynamics (14) is GES. \( \square \)

**Remark 1** Note that the globalness is given with respect to the chosen coordinate frame. It is not topologically possible to obtain results that are global in \( SO(3) \) using any coordinate frame of \( SO(3) \) like the Euler angles, Euler parameters, Euler-Rodrigues parameters or similar. Due to the topological properties of \( SO(3) \) these representations will either have one singularity or two equilibrium points, something which precludes global results on \( SO(3) \). The results in this paper are thus only global in the chosen coordinate frame.

**IV. OUTPUT FEEDBACK CONTROL**

In this section, the controller is derived by employing the observer backstepping method[12]. The control objective is defined as: \( \Theta(t) \rightarrow \Theta(t), \psi(t) \rightarrow \psi(t) \) as \( t \rightarrow \infty \), where \( \Theta(t) \triangleq [\theta(t), \psi(t)]^T \) contains the bounded and continuously differentiable reference trajectories for pitch and heading.

The observer can be rewritten according to
\[ \dot{\tilde{y}} = J(\Theta) \tilde{v} + \lambda L \tilde{y} \]
\[ M \tilde{v} + D(\tilde{\nu} \dot{\nu} + U_r^c \dot{\nu} + \dot{u_c} C^2 \dot{\nu} + \eta^b \dot{\nu}^b + \lambda M J(\Theta) \tilde{y} \]

\[ \dot{\tilde{b}} = -\kappa \lambda T^{-1} \tilde{b} + \kappa \lambda^2 \tilde{b} \tilde{y} \]

which is desirable since the control vector \( \tau \) naturally evolves in the body-frame.

**A. Controller design**

In this section, the Euler angle symbol \( \Theta \) is omitted when it is used in a transformation matrix for notational simplicity. **Step 1:** We define the first error vector as follows: \( z_1 \triangleq \Theta - \Theta(t), \) where \( \Theta \triangleq [\theta, \psi]^T \). Then, computing the corresponding error dynamics by differentiating \( z_1 \) with respect to time and using (18a) gives
\[ \dot{z}_1 = T_{\Theta} [\omega^b - \omega^b(t)] + \lambda F L \tilde{y} \]

where
\[ F = \left[ \begin{array}{ccc} 0_1 & 0_1 & 0_1 \\ 0_1 & 0_1 & 0_1 \end{array} \right], \; \phi_{\Theta}(t) = T_{\Theta} \omega^b(t) \]

\[ \dot{\omega}^b = [\dot{q}, \dot{r}]^T, \; \omega^b(t) = [q^b(t), r^b(t)]^T \]

The second error vector is defined as
\[ z_2 \triangleq \tilde{v} - \alpha \]

T20-008
where $\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]^T$ is a vector of stabilizing functions, and $z_2 = [z_{2,1}, z_{2,2}, \ldots, z_{2,5}, z_{2,6}]^T$ is the velocity tracking error vector. Inserting for $\hat{\nu}$ in (20) into (19) yields

$$
\dot{z}_1 = T_\Theta (\alpha_\Theta - \omega^t_\Theta (t)) + T_\Theta F z_2 + \lambda^T F \hat{\eta} \tag{21}
$$

where $\alpha_\Theta = [\alpha_3, \alpha_4]^T$ contains the stabilizing functions for the actuated states: pitch and yaw. In order to render (21) a stable differential equation, we choose the stabilizing function $\alpha_\Theta$ to evolve according to

$$
\dot{\alpha}_\Theta = \omega^t_\Theta (t) - T_\Theta^{-1} K_p z_1 \tag{22}
$$

where $K_p \in \mathbb{R}^{2 \times 2}$ is a positive diagonal matrix. This results in the following expression for the $z_1$-dynamics

$$
\dot{z}_1 = -\varepsilon K_p z_1 + \varepsilon (T_\Theta F z_2 + \lambda^T F \hat{\eta}) \tag{23}
$$

Here we have multiplied with $\varepsilon > 0$ to increase the design flexibility in the controller.

**Step 2:** Proceeding with the $z_2$-dynamics

$$
M \dot{z}_2 = -D(|\dot{\nu}_r|) \dot{\nu}_r - U_c C_1 \dot{\nu}_r - \dot{u}_c C_2 \dot{\nu}_r - g + \tau + J^{-1} b + \lambda^2 M J^{-1} K \dot{\eta} - M \dot{\alpha} \tag{24}
$$

Using feedback linearization, we choose the control vector according to

$$
\tau = D(|\alpha_r|) \alpha_r + U_c C_1 \dot{\nu}_r + \dot{u}_c C_2 \dot{\nu}_r + g + M \dot{\alpha} - J^{-1} b - K_d z_2 + \varepsilon J T^T F T z_1 \tag{25}
$$

where $\alpha_r = \alpha - \dot{\tau}$, and $K_d > 0$ is a diagonal matrix. Notice that the controller $\tau$ is considered as a general vector in $\mathbb{R}^5$, omitting the fact that the second and third element are zero, i.e. $\tau_r = \tau_w \equiv 0$. This will be treated in Section IV-C.

**B. Controller analysis**

Inspired by the stability analysis approach presented in [13], we want to formulate the controller error dynamics and observer estimation error as a cascaded system. In order to achieve this, we define new controller error dynamics; the error between the desired state $(\Theta_r(t), \omega^t_\Theta (t))$ and the actual state $(\Theta, \omega^t)$, and not the estimated state $(\hat{\Theta}, \hat{\omega}^t)$, which is the natural result of the observer backstepping technique used in the controller design in Section IV-A. Hence, we define the following new error vectors

$$
z^a_1 \triangleq \Theta - \Theta_r(t), \quad z^a_2 \triangleq \nu - \alpha^a \tag{26}
$$

$$
z^a \triangleq [z^a_1, z^a_2]^T \tag{27}
$$

where $\alpha^a = \omega^t_\Theta (t) - T_\Theta^{-1} z^a_1$. With the new error states we will show that the controller error dynamics $z^a$ can be written in a cascade with the nominal observer dynamics $x_N$ as follows

$$
z^a = f_{z^a}(t, z^a) + g_{z^a}(t, z^a, x_N) \tag{28a}
$$

$$
\dot{x}_N = f_{x_N}(t, x_N) \tag{28b}
$$

where the perturbation $g_{z^a}(t, z^a, x_N)$ is to be defined. The error system (28b) represents the nominal observer error dynamics depicted in (15), which $x_N = 0$ is proven GES in the proof of Proposition 2. Recall, however, that Proposition 2 is only valid under A.2, which claims bounded velocities of the vehicle. In order to avoid circularity in the following stability analysis, we lift A.2 and replace it with the following assumption on bounded desired velocities, which is a common method in output feedback controller design, see e.g. [3].

**A.3** There exists a constant $V_d \in \mathbb{R}_+$ such that the desired velocity is bounded according to $V_d = sup_t \|\omega^t_\Theta (t)\|$.

Under A.3, we have that the perturbation vector $g^a_{z^a}$ to the nominal observer error dynamics (15) can be upper bounded according to

$$
\|g^a_{z^a}\| = \|\dot{u}_c C^a_2 (\nu - \dot{\nu}_r^e - \nu^e_\Theta + \nu^e_{\overline{\Theta}}) + \dot{u}_c C^a_2 (J^{-1} (\zeta^a + \alpha) - \dot{\nu}_r^e - \nu^e_\Theta + \nu^e_{\overline{\Theta}})
\leq \|\dot{u}_c|C^a_2|V_d + \|z^a_2\| + \nu_c K^M \|z^a_1\| + \lambda^\xi \|z^a_2\|
\quad + \varepsilon (T_\Theta^{-1} K_p T \|x_N\| + V_c + \lambda^2 \|x_C\|) \tag{28c}
$$

by using (20) and (22), and where $\xi = \lambda/\beta_c$. Hence, $g^a_{z^a}$ consists only on error variables and can be treated as a standalone perturbation of the nominal system (28).

In order to rewrite the control vector (25) so that it consists of the actual states and not the estimated states, we use that $\hat{\eta} = \eta - \hat{\eta}$ and $\dot{\nu} = \nu - \nu$ resulting in

$$z_1 = z^a_1 - F \hat{\eta}, \quad z_2 = z^a_2 - \nu \tag{29}
$$

Using (29), the controller (25) can be expressed as

$$
\tau = D(|\alpha^a - \dot{\tau}|) (\alpha^a - \dot{\tau}) + U_c C_1 \dot{\nu}_r + \dot{u}_c C_2 \dot{\nu}_r + g + M \dot{\alpha} - J^{-1} b - K_d z_2 + \varepsilon J T^T F T z_1 + g_{z^a_2} \tag{30}
$$

Here we have collected the signals involving the estimation error $x_N$ into the perturbation vector $g_{z^a_2}$ as follows

$$
g_{z^a_2} = \left( D|\xi T^{-1}_\Theta K_p T \|x_N\| \right) (\omega^t_\Theta - \xi T^{-1}_\Theta K_p T x_1)
- \dot{u}_c C_2 \nu_r + \lambda(U_c C_1 - \dot{u}_c C_2) (\xi x_2 - \lambda \xi x_2)
\quad + \nu_c K^M \lambda \xi T^{-1}_\Theta K_p T x_1
\quad + \lambda \xi J^{-1} x_3 + \varepsilon J T^T F T x_1 + \xi \lambda J^{-1} x_3 \lambda \xi T^{-1}_\Theta K_p T x_1
$$

where we have used that $\dot{u}_c C_2 \dot{\nu}_r = u_c C_2 \nu_r - \dot{u}_c C_2 \nu_r - \dot{u}_c C_2 \nu_r$. Inserting the modified control vector (30) into the actual vehicle dynamics (3) results in the following $z^a$-dynamics

$$
\begin{bmatrix}
\varepsilon \\ M \dot{z}^a_2
\end{bmatrix} =
-\left[
\begin{bmatrix}
\varepsilon K_p \\ \varepsilon J T^T F T (K_d + \delta c) (\cdot) \\
\end{bmatrix}
\right] - \left[
\begin{bmatrix}
\varepsilon_{2x1} \\ \varepsilon_{2x2} (\cdot)
\end{bmatrix}
\right]
+ \left[
\begin{bmatrix}
\varepsilon_{2x1} \\ \varepsilon_{2x2} (\cdot)
\end{bmatrix}
\right] \tag{31}
\right)
$$

where the perturbation vector capturing the terms involving the estimation error becomes $g_{z^a_2} = \lambda \xi \left( \xi x_5 + \lambda \xi x_3 \right)$. Having established this, it follows that the rewritten tracking error state (27) can be viewed as a cascade with the nominal observer error dynamics (28b), and where the perturbation vector is given by $g_{z^a}(t, z^a, x_N) = \left[\begin{bmatrix} T^T \varepsilon_{2x1} \\ T^T \varepsilon_{2x2} (\cdot) \end{bmatrix}\right]$. Equivalent to the observer error dynamics in Section III-B, we notice that the (28b) includes the time-varying signals $(x(t), y(t), z(t))$, while we only consider the tracking error $z^a$. We thus consider $(x(t), y(t), z(t))$ as general time-varying signals using forward completeness, see e.g. [16] for details.

**Proposition 3** The origin $z^a = 0$ is U GAS under A.3 and if condition (16) is satisfied.
**Proof.** Inspired be the method described in [13], we resort to cascaded systems theory in the upcoming stability analysis. Each of the functions in (28) are analyzed separately in three steps:

1: The origin $x_N = 0$ of the nominal observer error dynamics (15), denoted as $\dot{x}_N = f_{x_N}(t, x_N)$ in (28), is proven GES in Section III-C.

2: At this step we want to establish the stability properties of the unperturbed system $\dot{z}^o = f_z(t, z^o)$ in (28a). Similarly as in Section III, we want to exploit the dissipative property of the hydrodynamic damping. We thus include the current estimation error in the stability analysis and propose the following radially unbounded Lyapunov function candidate

$$V_c(\zeta, \xi) = \frac{1}{2} \xi^T M \zeta + \frac{1}{2} \zeta^T \xi$$

where $\xi = z^o - h\nu_2$ and $\zeta = \xi^T [\zeta^T \zeta^T]^T$. Differentiating (32) with respect to time and inserting for the tracking dynamics (31) without the perturbation gives

$$\dot{V}_c(\zeta, \xi) = -\beta(\|\zeta\|^2) + g_c(\|\zeta\|)\sigma(\|\zeta(t_0)\|, t-t_0)$$

where $\beta > 0$, $g_c : \mathbb{R}_+ \to \mathbb{R}$, and $\sigma$ is a class $\mathcal{KL}$ function satisfying $\int_0^\infty \sigma(r,s)ds \leq \sigma_\infty r$, where $r > 0$ and $\sigma_\infty$ is some constant. In this case, since $x_C = 0$ is proven UGES and satisfying $x_C \leq W_c$, for all $t > t_0$, it can be shown that condition (34) holds, and thus $\zeta = 0$. This results in the perturbation being satisfied. Consequently, based on Theorem 1, this completes the proof.

3: This step involves the determination of the perturbation vector $g_z(t, z^o, x_N)$. Using P2, P5 and P4 gives

$$g_z^o \leq \xi(L_2 + 1)\|x_1\| + D_P(\|x_1\|) + \lambda \zeta \xi(\|x_1\|) + \lambda \|x_2\|$$

where $D_P(\|x_1\|)$ is a positive scalar function collecting the terms involved in the damping matrix. The perturbation can be shown to be upper bounded by the tracking error state and the estimation error as follows

$$\|g_z(t, z^o, x_N)\| \leq \theta_1(\|x_N\|) + \theta_2(\|x_N\|, \|z^o\|)$$

where $\theta_1, \theta_2 : \mathbb{R}_+ \to \mathbb{R}_+$. Thus, the linear growth restriction on $\|z^o\|$ in the perturbation is satisfied. Consequently, based on the three prior steps, it follows that the origin $z^o = 0$ and $x_N = 0$ of the system (31) and (15) is UGAS [15, Theorem 2.8]. This completes the proof.

Up to this point, we have only considered the nominal observer error dynamics (14). The following theorem establishes asymptotic stability of the overall output feedback controller. Let the error vector $x_t \in \mathbb{R}^{22}$ denote the complete error state excluding the current estimation error, i.e.

$$x_t \triangleq [\eta^T, \dot{\nu}^T, \delta^T, z_1^T, z_2^T]^T$$

**Theorem 1** The origin $x_t = 0$ and $x_C = 0$ of the cascaded system (7), (12) and (31) is UGAS under A.1 and A.3, and if (16) is satisfied.

**Proof.** We write the overall system including the current error dynamics in the following compact form

$$\begin{align*}
\dot{x} &= f_t(t, x_t) + g_t(t, x_t, x_C) \\
\dot{x}_C &= f_2(t, x_C)
\end{align*}$$

where $g_t(t, x_t, x_C) = [g_p^T, \theta^T_{7 \times 1}]^T$. The unperturbed system $\dot{z} = f_j(t, x_j)$ in (36a) is proven UGES in Proposition 3. Furthermore, the current estimation error dynamics (36b) are proven UGES in Section III-B. Following cascaded systems theory, it remains to show bounded growth on the perturbation vector $g_t(t, x_t, x_C)$.

It follows from (28c) that it can be bounded linearly by $\|x_1\|$ and hence satisfying

$$\|g_t(t, x_t, x_C)\| \leq \theta_3(\|x_C\|) + \theta_4(\|x_C\|, \|x_t\|)$$

The linear growth restriction on $\|x_1\|$ in the perturbation term is satisfied. It thus follows from [15] that the origin $x_t = 0, x_C = 0$ of the systems (7), (12) and (31) is UGAS. □

**C. Underactuated vehicle**

In this section we consider the fact that the vehicle is underactuated. Recalling that the control vector yields $\tau = [\tau_u, 0, 0, \tau_w, \tau_r]^T$. This results in a dynamic constraint in the controller for the unactuated states, i.e. sway and heave since we cannot assign control force in sway or heave, i.e. $\tau_w = \tau_u \equiv 0$. However, we will, by analyzing the dynamics of the controller, show that the sway and heave velocities converge to a bounded set due to the hydrodynamic damping which is present in all degrees of freedom. This method is inspired by [8]. From (25) we have that

$$\tau = D(|\alpha_r|)\alpha + U_C C_1 \dot{v}_r + \ddot{w} C_2 \dot{v}_r + \bar{M}$$

$$+ \underline{g} - J^{-1} \tilde{b} - K_2 z_2 - \varepsilon_1 J^T \chi^T z_1$$

where the bounded and converging variables are concatenated in the function $f_{\bar{a}}(\cdot)$ and are shown to be bounded or converging to zero except the bias term $J^{-1} \tilde{b}$. We will in what follows assume that the bias $\tilde{b}$ is bounded. Another common approach in marine applications is to define the bias as constant, i.e. $b = 0$ [6], which clearly manifests this assumption. We proceed by analyzing the $[\alpha_2, \alpha_3]$-dynamics by rewriting (38) in compact form and collecting all the bounded and converging signals into the vector function $f_{\bar{a}}(\cdot) \in \mathbb{R}^8$.
where \( \vec{a}_r \triangleq [\alpha_2 - \vec{v}_c, \alpha_3 - \vec{w}_c]^T \) and
\[
\begin{bmatrix}
\bar{M} & 0 \\
0 & \bar{M}
\end{bmatrix},
\begin{bmatrix}
D(\vec{a}_r) = \\
0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
\bar{Y} = \\
\bar{F}
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

The vector \( \bar{Y} \) captures the off-diagonal terms in the mass, damping and Coriolis matrices (\( \bar{C}_1 \) and \( \bar{C}_2 \)). Furthermore, notice that \( \bar{Y} \) consists solely of bounded and converging signals (\( \alpha_i, \vec{u}_c \)). Hence, we add \( \bar{Y} \) to the function \( f_\alpha(\cdot) \). It can be shown that the \( \vec{a}_r \) - subsystem is input-to-state stable from \( f_\alpha \) to \( \vec{a}_r \), due to hydrodynamic damping in all degrees of freedom. See [8] and [16] for more details on this approach.

### V. Case Study: The Minesniper MkII

The vehicle is torpedo shaped and is 1.93m long, 0.17m in diameter and its weight is 40kg. Surge and yaw control is provided by two thrusters located at the center on each side of the hull, and pitch control is done by movement of a mass along the x-direction of the vehicle, see e.g. [17]. The sea trials of Minesniper MkII were performed in Trondheimsfjorden near by Stjørdal, Norway. Figure 1 presents the tracking results in pitch and heading and the actuator action. The tracking performance is satisfactory with relatively small deflections of the actuators. The observer provides satisfactory estimates for the entire run with small deviations and little noise, see Fig. 2. Towards the end of the run, we experienced increased noise in the acoustical measurements mainly due to the topography of the seabed blocking the view of the acoustical receivers. Nevertheless, the observer seemed to cope with measurement drop outs and the noise in a satisfactory manner, see Fig. 3. Figure 3 shows the measured and the estimated position, depth and orientation of the Minesniper MkII. The waypoint watch radius was set to 5m. This caused the vehicle to slightly miss the third and the fourth way-point. Moreover, the reference system was chosen relatively slow in order to keep the yaw rate small. The main reason for this was that given the center location of the thrusters, several sea trials revealed that erratic usage of the thrusters could cause intractable yaw motion and even instability. Therefore, emphasis was placed on keeping the yaw motion within some relatively restrictive boundaries.

### VI. Conclusions

An output feedback controller was proposed for a slender body underwater vehicle. The CPM scheme consisted of two separate models, a 5 DOF vehicle model and a 3 DOF model.
accounting for the main current loads. Part of the vehicle CPM was linearized about the relative surge velocity. The nonlinear Luenberger observers and the controller, which was designed using the observer backstepping technique, were proven UGAS using Lyapunov and cascaded system theory. Experimental sea trials performed on the Minesniper MKII demonstrated satisfactory observer and tracking performance.

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