Output Feedback Decoupling of Neutral Time Delay Systems

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Abstract—The problem of Input Output Decoupling is studied for the case of general neutral multi-delay systems, via proportional realizable output feedback. Using a pure algebraic approach, the necessary and sufficient conditions for the problem to have a realizable solution are established and the general analytical expressions of the proportional realizable controller matrices, are derived.

I. INTRODUCTION

In the frequency domain the forced response of the general class of linear neutral multi-delay differential systems is governed by the following algebraic system of equations [18]

\[ sX(s) = A(e^{sT})X(s) + B(e^{sT})U(s) \quad (1a) \]
\[ Y(s) = C(e^{sT})X(s) \quad (1b) \]

where \( X(s) \) denotes the Laplace transform of the state vector \( x(t) \in \mathbb{R}^n \), \( U(s) \) denotes the Laplace transform of the control input vector \( u(t) \in \mathbb{R}^m \) and \( Y(s) \) denotes the Laplace transform of the performance output vector \( y(t) \in \mathbb{R}^r \). The elements of the system matrices in (1), namely the elements of the matrices \( A(e^{sT}), B(e^{sT}), \) and \( C(e^{sT}), \) are multivariable rational functions of \( e^{-\tau_1}, \ldots, e^{-\tau_q} \) (or more compactly of \( e^{sT} = \exp(-s\tau_1) \cdots \exp(-s\tau_q) \), where \( \exp(\cdot) = e^{\cdot} \) denotes the exponential of the argument quantity). \( \mathbb{R}_+ \) denotes the set of these multivariable rational functions. The quantities \( \tau_i \ (i = 1, \ldots, q) \) are positive real numbers denoting point delays. Without loss of generality the delays \( \tau_1, \ldots, \tau_q \in \mathbb{R} \) are considered to be rationally independent, namely linearly independent among themselves over the field of rational numbers, i.e. there are no rational numbers (dependence coefficients) expressing one delay as a linear combination of the others. For the special case of rationally dependent delays where all delays are multiple of one, with dependence coefficients being integers, the delays are called commensurate [6].

Note that if the elements of \( A(e^{sT}), B(e^{sT}), \) and \( C(e^{sT}), \) are multivariable polynomial functions of \( e^{-\tau_1}, \ldots, e^{-\tau_q}, \) the special case of general retarded multi-delay systems is derived.

Here, the design goal is that of Input/Output (I/O) Decoupling, namely to derive a closed loop system in which each output is controlled only by one external input. The output feedback is proposed to be of proportional type, i.e.

\[ U(s) = K(e^{sT})Y(s) + G(e^{sT})\Omega(s) \quad (2) \]

where \( \Omega(s) \) is the \( m \times 1 \) vector of external inputs. The elements of the matrices \( K(e^{sT}), \) and \( G(e^{sT}), \) are multivariable rational functions of \( e^{-\tau_1}, \ldots, e^{-\tau_q} \) (or more compactly of \( e^{sT} \)) namely they belong to the same set as the elements of the system matrices of system (1).

Substituting the control law (2) to system (1) the problem of I/O decoupling via output feedback is formally stated as in the following equation

\[ C(e^{sT})[sI_r - A(e^{sT}) - B(e^{sT})K(e^{sT})C(e^{sT})]^{-1} \times B(e^{sT})G(e^{sT}) = \text{diag}\{h_i(s, e^{sT})\} \quad (3) \]

where \( h_i(s, e^{sT}) \neq 0 \ (i = 1, \ldots, m ) \) is the \( i \)-th diagonal element of the decoupled closed-loop transfer function which (according to (3)) is a rational function of \( s \) with numerator and denominator polynomials having coefficients being multivariable rational functions of \( e^{-\tau_1}, \ldots, e^{-\tau_q} \) (or more compactly of \( e^{sT} \)).

Note that even though the elements of the matrices \( A(e^{sT}), B(e^{sT}), \) and \( C(e^{sT}), \) are not restricted to be realizable rational functions of \( e^{-\tau_1}, \ldots, e^{-\tau_q}, \) the implementability of the controller requires that the
elements of the matrices $K(e^{s\tau})$ and $G(e^{s\tau})$ must be realizable i.e.
\[
\lim_{s \to 0^+} K(e^{s\tau}) : \text{finite}, \quad \lim_{s \to 0^+} G(e^{s\tau}) : \text{finite}.
\]

The problem of input-output (I/O) decoupling of time delay systems has attracted considerable attention for the special case of commensurate retarded delay systems in [10], [4]-[5] and [11]-[15]. In [10] the problem has been studied for the first time with controllers involving possibly predictions. In [4], certain conditions have been proposed for the solution problem via a controller not involving predictions. In [11] the problem is studied via predictors for the case of time varying commensurate systems. In [12] (unimodular precompensator) and [13] the problem of I/O decoupling is studied together with the problem of coefficient assignment and under certain controllability conditions, via proportional and/or dynamic feedback. The sufficient conditions, established in [5], [14] and [15] via a controller not involving predictors, are essentially wider than the previous conditions in the field. The assumptions made in [5], [14] and [15] are: the biproperness of the precompensator and the investigation of the realizability of a feedback in a special class of the controller matrices (possibly involving predictors) that solve the I/O decoupling problem. Furthermore in [5] the problem of stability of the proportional controllers has been studied. For the case of the decoupling problem for standard neutral commensurate systems of standard form, some first results have been presented in [6]. In [7] and [8] the I/O decoupling problem is studied for normal systems with system and controller matrices having their elements in a Noetherian ring. Except [10] and [11], where a controller involving possibly predictions is proposed, all results in the field focus towards a realizable controller, namely a controller not involving predictions. System analysis and control design problems of neutral time delay systems attract the interest of many researchers e.g. [3] and [2]. In [1] the problem of decoupling with simultaneous disturbance rejection has been studied for general neutral multi-delay differential systems via realizable state feedback. All results in the field focus towards a realizable controller that is a controller not involving predictions. For the case of output feedback the only results regarding retarded time delay systems have been presented in [9]. The contribution of the present paper consists in establishing the following aspects, using a proportional realizable output feedback law: The necessary and sufficient conditions for the solvability of the decoupling problem for general neutral multi-delay systems and the general solution of the respective output feedback controllers.

II. PRELIMINARIES

A. From Multi-Delays to Multivariable Rational Functions

Let $z = [z_1, \ldots, z_q]$, where $z_i \in \mathbb{C}$ ($i = 1, \ldots, q$). Let $\mathbb{R}_o(z)$ be the set (field) of multivariable rational functions of $z_1, \ldots, z_q$ (or more compactly of $z$) with real coefficients. Let $\gamma(z) = \sum_{\chi \in \mathbb{R}_o(z)} \gamma_\chi z_1^{\chi_1} \cdots z_q^{\chi_q}$ be a multivariable polynomial. Let $\mathbb{R}_e(e^{s\tau})$ be the set of multivariable rational functions $e^{-\tau_1}, \ldots, e^{-\tau_q}$ (or more compactly of $e^{s\tau}$). If the delays $\tau_1, \ldots, \tau_q$ are rationally independent, then there is a one to one correspondence between the elements of $\mathbb{R}_o(z)$ and $\mathbb{R}_e(e^{s\tau})$. The two sets are isomorphic in the sense that if two quantities belonging to one of these fields are equal then so are the respective quantities in the other field.

It is noted that the set of exponents of the polynomial $\gamma(e^{s\tau})$ with respect to the variables $e^{-\tau_1}, \ldots, e^{-\tau_q}$, is $\mathbb{R}\{\gamma(z)\}$.

It is important to mention that the range of exponents of a polynomial of $e^{s\tau}$ with respect to $e^{-\tau}$, is denoted by $\exists (\cdot)$. So, the range of exponents of $\gamma(e^{s\tau})$, with respect to $e^{-\tau}$, is defined as follows: $\exists \{\gamma(e^{s\tau})\} = \{\varphi \in \mathbb{R} / \varphi = \sum_{i=1}^{q} \kappa_i \tau_i ; [\kappa_1, \ldots, \kappa_q] \in \mathbb{N}\{\gamma(z)\}\}$. Clearly $\exists \{\gamma(e^{s\tau})\}$ is a finite set.

Let $\mathbb{R}_o(s, z)$ be the set of rational functions of $s$ with coefficients in $\mathbb{R}_o(z)$. This set is also a field.

Also, consider the set $\mathbb{R}_e(s, e^{s\tau})$, namely the set of rational functions of $s$ with coefficients in $\mathbb{R}_e(e^{s\tau})$, where as already mentioned in Section 1 the delays $\tau_1, \ldots, \tau_q$ are rationally independent. Since the delays $\tau_1, \ldots, \tau_q$ are rationally independent, there is also a one to one correspondence between the elements of $\mathbb{R}_e(s, e^{s\tau})$ and $\mathbb{R}_o(s, z)$. These two sets are isomorphic in the sense that if two quantities belonging to one of these fields are equal then so are the respective quantities in the other field.

B. Realizability of transformations

A multivariable rational function of $\exp(-s\tau_1), \ldots, \exp(-s\tau_q)$ (i.e. a rational function that belongs to $\mathbb{R}_e(e^{s\tau})$) is said to be realizable ([6], [12]) if no predictors are required for its realization. This is formally stated as: the limit of the rational function of $e^{s\tau}$, for $s$ being a positive real number and for $s$ tending to infinity, is finite. The set of multivariable realizable
rational functions of $e^{sT}$ is denoted by $\mathbb{R}_r \{ e^{sT} \}$. Clearly $\mathbb{R}_r \{ e^{sT} \}$ is a ring.

**Lemma 2.1** [17]: Let $N(e^{sT})$ be of full row rank over $\mathbb{R}_r \{ e^{sT} \}$, then it holds that: there exist two invertible matrices, let $M(e^{sT})$ and $\Xi(e^{sT})$, with $\Xi(e^{sT})$ bi-realizable, i.e. $\Xi(e^{sT})$ is realizable, i.e.

$$\lim_{s \to \infty} \Xi(e^{sT}) : \text{finite}$$

and $[\Xi(e^{sT})]^{-1}$ is realizable

$$\lim_{s \to \infty} [\Xi(e^{sT})]^{-1} : \text{finite},$$

having the property

$$M(e^{sT}) N(e^{sT}) \Xi(e^{sT}) = [I_r \ 0]. \quad (4)$$

Explicit formulae for $M(e^{sT})$ and $\Xi(e^{sT})$ are given in [17].

The above transformation, being of great importance for the study of realizability issues for time delay systems, has been called right bi-realizable unitarizing transformation [17].

The conditions for the existence of realizable matrices in the range of a non-homogeneous map with elements in $\mathbb{R}_r \{ e^{sT} \}$, will be presented. Consider the subspace defined be the affine transformation

$$f(e^{sT}) = \psi(e^{sT}) N(e^{sT}) - \nu(e^{sT}) \quad (5)$$

where $\psi(e^{sT})$ is an arbitrary $1 \times a$ row vector, $N(e^{sT})$ is an $a \times \beta$ fixed full row matrix over $\mathbb{R}_r \{ e^{sT} \}$ and $\nu(e^{sT})$ is a fixed $1 \times \beta$ row vector with elements in $\mathbb{R}_r \{ e^{sT} \}$. The elements of $N(e^{sT})$, $\nu(e^{sT})$ and $\psi(e^{sT})$ belong to $\mathbb{R}_r \{ e^{sT} \}$.

To present the conditions for the linear map (5) to be realizable, define

$$\nu'(e^{sT}) = \nu(e^{sT}) \Xi(e^{sT}) [0 \ 1 \ 0]$$

$$\nu''(e^{sT}) = \nu(e^{sT}) \Xi(e^{sT}) [I_{\beta \times a}]$$

**Theorem 2.1** [17]: There exist a vector $\psi(e^{sT})$ such that the linear map $f(e^{sT}) = \psi(e^{sT}) N(e^{sT}) - \nu(e^{sT})$ is realizable, with $N(e^{sT})$ being of full row rank over the field $\mathbb{R}_r \{ e^{sT} \}$, if and only if the vector $\nu'(e^{sT})$ is realizable, i.e.

$$\lim_{s \to \infty} [\nu'(e^{sT})] : \text{finite}.$$

**Corollary 2.1** [17]: If Theorem 2.1 holds, then the general form of $\psi(e^{sT})$ yielding realizable $f(e^{sT})$ is

$$\psi(e^{sT}) = [\lambda(e^{sT}) + \nu''(e^{sT})] M(e^{sT}),$$

where $\lambda(e^{sT}) \in [\mathbb{R}_r \{ e^{sT} \}]^{\beta \times a}$ is an arbitrary realizable vector. The general form of the realizable $f(e^{sT})$, let $f(e^{sT})$, is given by the relation

$$f(e^{sT}) = [\lambda(e^{sT}) \big] [\nu'(e^{sT})] [\Xi(e^{sT})]^{-1}. \quad (5)$$

**III. SOLUTION OF THE DECOUPLING PROBLEM VIA AN OUTPUT FEEDBACK CONTROLLER INVOLVING POSSIBLY PREDICTORS**

Let $A(z)$, $B(z)$, $C(z)$, $K(z)$ and $G(z)$ be the corresponding matrices (with elements in $\mathbb{R}_n \{ z \}$) of the matrices $A(e^{sT})$, $B(e^{sT})$, $C(e^{sT})$, $K(e^{sT})$ and $G(e^{sT})$ (with elements in $\mathbb{R}_r \{ e^{sT} \}$). Also let $\{ h_i(\{ s, e^{sT} \}) \}$ be the corresponding matrix (with elements in $\mathbb{R}_n \{ s, e^{sT} \}$) of the matrix $\text{diag} \{ h_i(\{ s, e^{sT} \}) \}$ (with elements in $\mathbb{R}_r \{ s, e^{sT} \} \}$. Then, based on the isomorphism between $\mathbb{R}_r \{ s, e^{sT} \}$ and $\mathbb{R}_n \{ s, e^{sT} \}$, equation (3) formulating the I/O decoupling problem, is reduced to the following matrix equation

$$C(z) [sI_{\beta} - A(z) - B(z) K(z) C(z)^{-1} B(z) G(z)] = \text{diag} \{ h_i(\{ s, e^{sT} \}) \} \quad (7)$$

According to [1], the necessary and sufficient condition for the Decoupling problem of general neutral multi-delay systems to have a solution, via a proportional state feedback controller (possibly involving predictions), is:

$$\det \left[ C^*(z) B(z) \right] \neq 0 \quad (8)$$

where

$$C^*(z) = \begin{bmatrix} c_1(z) [A(z)] \delta^1 \\ \vdots \\ c_m(z) [A(z)] \delta^m \end{bmatrix},$$

$$\delta = \min \left\{ j : c_i(z) [A(z)]^j B(z) \neq 0, \right\} \in \{ 0, 1, \ldots, n - 1 \}.$$
with \( c_i(z) \) denoting the \( i\)-th row of \( C(z) \). The present case namely the output feedback case is a special case of the respective state feedback case. Hence, the class of all solutions of \( G(z) \) is a subset of the respective general solution of \( G(z) \) for the state feedback case expressed by the formula

\[
G(z) = \left[ C^*(z) B(z) \right]^{-1} \left[ \text{diag} \{ p_i(z) \} \right]^{-1} \quad (9)
\]

where \( p_i(z) \in \mathbb{R}_0(z), i = 1, \ldots, m \), are arbitrary nonzero parameters. Let \( H(s,z) = C(z) [sI - A(z)]^{-1} B(z) \). The condition (8) implies the invertibility of the open loop system, namely the invertibility of the matrix \( H(s,z) \).

Also, let \( R(s,z) = \left[ \text{diag} \{ s^k+1 \} H(s,z) \right]^{-1} \) and \( \eta_i(s,z) = [p_i(z) h_i(s,z)]^{-1} s^{-k-1} \). Then the problem formulation can be reduced to (9) as well as the equation:

\[
\left[ C^*(z) B(z) \right]^{-1} \text{diag} \{ \eta_i(s,z) \} = R(s,z) - K(z) \text{diag} \{ s^{-k-1} \} \quad (10)
\]

The matrix \([R(s,z)]\) can be expanded in negative power series of \( s \) to yield

\[
[R(s,z)] = R_0(z) s^0 + R_1(z) s^{-1} + R_2(z) s^{-2} + \cdots \quad (11)
\]

where \( R_0(z) = \left[ C^*(z) B(z) \right]^{-1} \). Also the scalars \( \eta_i(s,z) \) may be expanded in negative series of \( s \) to yield

\[
\eta_i(s,z) = 1 + \eta_{i1}(z) s^{-1} + \eta_{i2}(z) s^{-2} + \cdots \quad (12)
\]

Finally, let \( R^*(z) \) be an \( m \times m \) matrix of the form

\[
R^*(z) = [R^*_0(z) \cdots R^*_m(z)] \quad \text{with } R^*_i(z) = R_{i+1}(z) e_i,
\]

where, \( e_i \) is the \( m \times 1 \) unity vector having the unity in its \( i\)-th position.

**Lemma 3.1:** The Input Output Decoupling problem for the system \((A(z), B(z), C(z))\) is solvable, via a proportional output feedback controller \((K(z), G(z))\), if and only if the following conditions are satisfied:

\begin{enumerate}
  \item \( \det \left[ C^*(z) B(z) \right] \neq 0 \)
  \item \( \left[ C^*(z) B(z) R^*_i(z) \right]_{k,\lambda} = 0, \quad (k \neq \lambda; \quad k = 1, \ldots, m; \quad j = 0, \ldots, d_i d_j + 2 \cdots 2n; \quad \lambda = 1, \ldots, m) \), where \( \bullet_{k,\lambda} \) denotes the \((k,\lambda)\)-th element of the argument matrix.
\end{enumerate}

**Proof:** As already mentioned the condition (i) is a necessary condition for the problem to be solvable. After expanding both sides of equation (10) in power series of \( s \) and after equating equal powers of \( s \) in both sides of the equation, it can readily be observed that (10) is equivalent to the condition (ii) and the equation

\[
\left[ C^*(z) B(z) \right]^{-1} \text{diag} \{ \eta_{i+1} \} = R^*(z) - K(z). \quad \text{The latter equation can be considered as the respective general solution for } K(z). \quad \text{Thus, the proof has been completed.} \quad \blacksquare
\]

According to the proof of the above theorem the general solution of the controller matrices is \( G(z) = \left[ C^*(z) B(z) \right]^{-1} \text{diag} \{ p_i(z) \} \) and \( K(z) = R^*(z) - \left[ C^*(z) B(z) \right]^{-1} \text{diag} \{ \eta_{i+1} \} \) where \( p_i(z) \in \mathbb{R}_0(z), (i = 1, \ldots, m) \) are arbitrary nonzero parameters and where \( \eta_{i+1}(z) \in \mathbb{R}_0(z), (i = 1, \ldots, m) \) are arbitrary parameters.

Based on the above and the isomorphism between \( \mathbb{R}_0(z) \) and \( \mathbb{R}_r(e^{-s}) \) as well as the isomorphism between \( \mathbb{R}_0(s,z) \) and \( \mathbb{R}_s(s,e^{-s}) \) the following theorem can readily be established.

**Theorem 3.1:** The Input Output Decoupling problem for general neutral multi-delay systems is solvable, via a proportional output feedback controller involving possibly nonzero predictors, if and only if the following conditions are satisfied:

\begin{enumerate}
  \item \( \det \left[ C^*(e^{sT}) B(e^{sT}) \right] 
eq 0 \)
  \item \( \left[ C^*(e^{sT}) B(e^{sT}) R^*_i(z) \right]_{k,\lambda} = 0, \quad (k \neq \lambda; \quad k = 1, \ldots, m; \quad j = 0, \ldots, d_i d_j + 2 \cdots 2n; \quad \lambda = 1, \ldots, m) \), where \( \bullet_{k,\lambda} \) denotes the \((k,\lambda)\)-th element of the argument matrix.
\end{enumerate}

The general solution of the proportional output feedback controller matrices, involving possibly predictors, is

\[
G(e^{sT}) = \left[ C^*(e^{sT}) B(e^{sT}) \right]^{-1} \text{diag} \{ p_i(e^{sT}) \} \quad (13)
\]

\[
K(e^{sT}) = R^*(e^{sT}) - \left[ C^*(e^{sT}) B(e^{sT}) \right]^{-1} \text{diag} \{ \eta_{i+1}(e^{sT}) \} \quad (14)
\]

where \( p_i(e^{sT}) \in \mathbb{R}_r(e^{sT}), (i = 1, \ldots, m) \) are arbitrary nonzero parameters and where \( \eta_{i+1}(e^{sT}) \in \mathbb{R}_r(e^{sT}), (i = 1, \ldots, m) \) are arbitrary parameters. \( \blacksquare \)

For the special case where the system matrices and the controller matrices are independent from the delays the Theorem 3.1 can be considered as an alternative expression of the solution of the input output decoupling problem of linear time invariant systems via proportional output feedback presented in [16].

**IV. SOLUTION OF THE I/O DECOUPLING PROBLEM VIA REALIZABLE PROPORTIONAL OUTPUT FEEDBACK**

Let \( N^{(i)}(e^{sT}) = -e_i^T \left[ C^*(e^{sT}) B(e^{sT}) \right]^T \quad i = 1, \ldots, m \).

Note that \( N^{(i)}(e^{sT}) \) is a nonzero row vector. Consider the right birealizable unitarizing transformation of the row
Let the scalar \( M_i(\varepsilon^T) \) and the \( m \times m \) birealizable matrix \( \Xi_i(\varepsilon^T) \) denote the matrices of the right birealizable unitarizing transformation of the row vector \( N_i(\varepsilon^T) \) i.e. the matrices for which

\[
M_i(\varepsilon^T) N_i(\varepsilon^T) \Xi_i(\varepsilon^T) = [1 \quad 0_{m-1}].
\]

Now we are in position to establish the following theorem.

**Theorem 4.1.** The Input Output Decoupling problem for general neutral multi-delay systems is solvable, via a proportional realizable output feedback controller, if and only if the following conditions are satisfied:

(i) \( \det \left[ C^* (\varepsilon^T) B (\varepsilon^T) \right] \neq 0 \)

(ii) \( \left[ C^* (\varepsilon^T) B (\varepsilon^T) R_i (\varepsilon^T) \right]_{k \lambda} = 0, \quad (\kappa = \lambda; \quad k = 1, \ldots, m; \quad j = 0, \ldots, d_i - 2 \times 2m) \), where \( \bullet_{1 \times \lambda} \) denotes the \((\kappa, \lambda)\)-th element of the argument matrix.

(iii) \( \gamma_i R_{i+1}^T (\varepsilon^T) \Xi_i (\varepsilon^T) \begin{bmatrix} 0_{m-1} \\ 1_{m-1} \end{bmatrix}, \quad (i = 1, \ldots, m) \)

is realizable.

**Proof:** As already mentioned the condition (i) is a necessary condition for the problem to be solvable. After expanding both sides of the design equation (10) in power series of \( s \), it can readily be observed that (10) is equivalent to the condition (ii) as well as the equation

\[
\left[ C^* (\varepsilon^T) B (\varepsilon^T) \right]^{-1} \left[ \text{diag} \left( \eta_{i+1} (\varepsilon^T) \right) \right] = R_i (\varepsilon^T) - K (\varepsilon^T)
\]

Dividing the latter equation in columns we get

\[
\left[ C^* (\varepsilon^T) B (\varepsilon^T) \right]^{-1} \gamma_i \eta_{i+1} (\varepsilon^T) = R_{i+1} (\varepsilon^T) \gamma_i - k_i (\varepsilon^T)
\]

where \( k_i (\varepsilon^T) \) is the \( i \)-th column of the matrix \( K (\varepsilon^T) \). For the latter equation to be satisfied via a realizable \( k_i (\varepsilon^T) \) an appropriate \( \eta_{i+1} (\varepsilon^T) \) should exist, such that for \( (i = 1, \ldots, m) \) holds

\[
R_{i+1} (\varepsilon^T) \gamma_i - k_i (\varepsilon^T) = \gamma_i \eta_{i+1} (\varepsilon^T) N_i (\varepsilon^T): \text{realizable (15)}
\]

where use was made of the definition of \( N_i (\varepsilon^T) \) in the beginning of this section.

Applying Theorem 2.1 to the linear map (15) and upon using the definitions just before Theorem 4.1 the condition (iii) is derived to be a necessary and sufficient condition for the existence of a realizable \( K (\varepsilon^T) \).

With regard to \( G(\varepsilon^T) \) and according to (13) a realizable solution is always guaranteed by appropriately choosing the arbitrary nonzero parameters \( p_i (\varepsilon^T) \).

Clearly, and upon using Corollary 2.1 the general solution of these parameters is

\[
\gamma_i R_{i+1} (\varepsilon^T) \Xi_i (\varepsilon^T) \begin{bmatrix} 0_{m-1} \\ 1_{m-1} \end{bmatrix}, \quad (i = 1, \ldots, m)
\]

where \( p_i (\varepsilon^T) \) (\( i = 1, \ldots, m \)) are arbitrary nonzero realizable scalars.

**Theorem 4.2:** If the conditions (i)-(iii) in Theorem 4.1 are satisfied, then the general analytic expressions of the proportional realizable output feedback controller matrices are:

\[
G_i (\varepsilon^T) = \left[ C^* (\varepsilon^T) B (\varepsilon^T) \right]^{-1} \gamma_i \eta_{i+1} (\varepsilon^T) N_i (\varepsilon^T) p_i (\varepsilon^T)
\]

\[
K_i (\varepsilon^T) = \rho_i (\varepsilon^T) - \left[ C^* (\varepsilon^T) B (\varepsilon^T) \right]^{-1} \gamma_i \eta_{i+1} (\varepsilon^T) M_i (\varepsilon^T)
\]

where \( \gamma_i (\varepsilon^T) = -\epsilon_i R_{i+1}^T (\varepsilon^T) \Xi_i (\varepsilon^T) \gamma_i \) and where the free parameters are the arbitrary realizable nonzero scalars \( p_i (\varepsilon^T) \) (\( i = 1, \ldots, m \)) and the arbitrary realizable scalars \( \lambda_i (\varepsilon^T) \) (\( i = 1, \ldots, m \)).

**Proof:** Taking advantage of the proof of Theorem 4.1 the general solution of the arbitrary parameter \( \gamma_i (\varepsilon^T) \) guaranteeing realizability of \( G(\varepsilon^T) \) is given by the relation

\[
\gamma_i (\varepsilon^T) = -\epsilon_i R_{i+1}^T (\varepsilon^T) \Xi_i (\varepsilon^T) \gamma_i + M_i (\varepsilon^T) p_i (\varepsilon^T).
\]

From this relation in (13) relation (16a) is derived.

Also, according to the proof of Theorem 4.1, \( k_i (\varepsilon^T) = \rho_i (\varepsilon^T) \gamma_i - \left[ C^* (\varepsilon^T) B (\varepsilon^T) \right]^{-1} \gamma_i \eta_{i+1} (\varepsilon^T) \)

If the condition (iii) of Theorem 4.1 is satisfied, then according to Corollary 2.1, the general solution of the free parameter \( \lambda_i (\varepsilon^T) \), guaranteeing realizability of \( k_i (\varepsilon^T) \), is

\[
\lambda_i (\varepsilon^T) = \left[ \gamma_i (\varepsilon^T) + \lambda_i (\varepsilon^T) M_i (\varepsilon^T) \right] \gamma_i (\varepsilon^T)
\]

Hence, the general realizable solution of \( k_i (\varepsilon^T) \) is

\[
k_i (\varepsilon^T) = \rho_i (\varepsilon^T) \gamma_i - \left[ C^* (\varepsilon^T) B (\varepsilon^T) \right]^{-1} \gamma_i \left[ \gamma_i (\varepsilon^T) + \lambda_i (\varepsilon^T) M_i (\varepsilon^T) \right] \gamma_i (\varepsilon^T)
\]

From the above expression the general solution in (16b) is derived.

**V. ILLUSTRATIVE EXAMPLE**

Consider the system of the form (1), where

\[
A(\varepsilon^T) = \begin{bmatrix} -e^{-cT} & 1 + (1 + e^{-cT})^{-1} - 2e^{-cT} & 0 \\ e^{-cT} & -1 - 2e^{-cT} & 0 \\ -e^{-cT} & 2e^{-cT} & -1 \end{bmatrix}
\]

\[
C(\varepsilon^T) = \begin{bmatrix} 1 + e^{-cT} & 2 + e^{-cT} & 0 \\ 0 & 2 & 1 \end{bmatrix}
\]
necessary and sufficient conditions for the existence of a solution have been established and the general analytical expressions of the proportional realizable controller matrices have been derived.

REFERENCES


