Hierarchical Control Implementation

Leonardo Cavarischia, Leonardo Lanari
Dipartimento di Informatica e Sistemistica “A. Ruberti”
Università di Roma “La Sapienza”
Via Eudossiana n. 18 - 00184 ROMA - ITALIA
{cavarischia,lanari}@dis.uniroma1.it

Abstract—The recently introduced concept of system abstraction appears to be a promising tool for control purposes in a hierarchical framework. While propagation of system’s properties has been extensively studied, little investigation has been carried out on how a high-level control law is implemented on the low-level original system. In this paper we address this problem from a geometric point of view.

I. INTRODUCTION

Model reduction techniques play a crucial role in the study of Large Scale Systems. Standard approaches (e.g. [2], [9]) are based on the definition of a reduced-order model which captures fundamental properties of the full-order one. Under certain conditions control design on the coarser model ensures achievement of control objectives on the original system with a known and bounded performance degradation [12]. A different approach is possible on the original system with a known and bounded performance degradation [12]. A different approach is possible on the original system with a known and bounded performance degradation [12]. A different approach is possible on the original system with a known and bounded performance degradation [12]. A different approach is possible on the original system with a known and bounded performance degradation [12]. A different approach is possible on the original system with a known and bounded performance degradation [12].

A. The recently introduced concept of system abstraction appears to be a promising tool for control purposes in a hierarchical framework. While propagation of system’s properties has been extensively studied, little investigation has been carried out on how a high-level control law is implemented on the low-level original system. In this paper we address this problem from a geometric point of view.

II. φ-RELATED LINEAR VECTOR FIELDS

Definition 2.1: (φ-related linear vector fields [1])
Consider the linear vector fields $\xi = \Lambda \xi$ and $\eta = \Theta \eta$ respectively on linear spaces $\mathcal{M} \subset \mathbb{R}^n$ and $\mathcal{N} \subset \mathbb{R}^m$, and the map $\eta = \Phi \xi$ from $\mathcal{M}$ to $\mathcal{N}$. Then $\Lambda$ and $\Theta$ are φ-related iff diagram of Fig. 1 commutes, i.e.

$$\Phi \Lambda = \Theta \Phi$$  

Moreover, if $\xi(t, \xi_0)$ is an integral curve of $\Lambda$, 

$$\eta(t, \eta_0) = \eta(t, \Phi \xi_0) = \Phi \xi(t, \xi_0)$$  

is an integral curve of $\Theta$. 

φ-related vector fields involve an invariant subspace 

• if $\Phi = \Phi$ is epic (full row rank) then $\Theta = \Lambda|_{\mathcal{M}/\mathcal{V}}$, where $\mathcal{V} = \text{Ker} \ C$ is an $\Lambda$-invariant, and $C \Lambda = \Lambda|_{\mathcal{M}/\mathcal{V}} C$

• if $\Phi = \Phi$ is monic (full column rank) then $\Lambda = \Theta|_{\mathcal{M}}$, where $\mathcal{M} = \text{Im} \ \Pi \subset \mathcal{N}$ is $\Theta$-invariant, and $\Pi \Theta|_{\mathcal{M}} = \Theta$.

II. CONTROL SYSTEMS PROJECTIONS/RESTRICTIONS

The above notions can be extended to linear control systems if thought of as input parametrized linear vector fields.

Definition 3.1: Given the linear control system 

$$\dot{x} = Ax + Bu \quad x \in \mathcal{X} \quad u \in \mathcal{U}$$

and a subspace $\mathcal{V} \subset \mathcal{X}$ then 

i) given $y = Cx$ s.t. $\text{Ker} \ C = \mathcal{V}$, system 

$$\dot{y} = Fy + Gu \quad y \in \mathcal{X}/\mathcal{V}$$

is a projection of $\Sigma_o$ on $\mathcal{X}/\mathcal{V}$ if each vector field $\Sigma_o$ corresponding to a fixed input value $u$ is the quotient of $\Sigma_o$ for the same $u$.

ii) Given $x = \Pi z$ s.t. $\text{Im} \ \Pi = \mathcal{V}$, system 

$$\dot{z} = Qz + Hu, \quad z \in \mathcal{V}$$


Fig. 1. φ-related linear vector fields
is a restriction of $\Sigma_o$ to $\mathcal{V}$ if each vector field $\Sigma_r$ corresponding to a fixed input value $\bar{u}$ is the restriction of $\Sigma_o$ for the same $\bar{u}$.

We recall the following Theorem from [5].

**Theorem 3.2:** Given the linear control system

$$\Sigma_o: \dot{x} = Ax + Bu \quad x \in \mathcal{X} \quad u \in \mathcal{U}$$

and an invariant subspace $\mathcal{V} \subset \mathcal{X}$, then

i) system

$$\Sigma_o \quad \dot{y} = Fy + Gu \quad y \in \mathcal{X}/\mathcal{V}$$

is a projection of $\Sigma_o$ on $\mathcal{X}/\mathcal{V}$ iff

$$F = A_{|\mathcal{X}/\mathcal{V}} \tag{3}$$

$$G = CB \tag{4}$$

with $\text{Ker } C = \mathcal{V}$.

ii) System

$$\Sigma_r: \dot{z} = Qz + Hu, \quad z \in \mathcal{V}$$

is a restriction of $\Sigma_o$ to $\mathcal{V}$ iff

$$Q = A_{|\mathcal{V}} \tag{5}$$

$$B = \Pi H \tag{6}$$

with $\text{Im } \Pi = \mathcal{V}$. Conditions (4) and (6) can also be obtained from the commutativity of diagram in Figure 2 and state that $\text{Im } G = (\text{Im } B + \mathcal{V})/\mathcal{V}$, while $H: \mathcal{U} \rightarrow \mathcal{V}$ has the action of $B$ with codomain reduced to $\mathcal{V}$.

The well-known notion of aggregation [2] is equivalent to projection (see also [7], [8], [5]). Aggregation is a model reduction technique since it allows to concentrate on a chosen subset $y = Cx$ of the state variables, also called aggregated variables. Aggregation defines equivalence classes within the state space while $A$-invariance of $\mathcal{V} = \text{Ker } C$ reveals that states within an equivalence class, corresponding to the linear variety $\mathcal{L} = \{x\} + \mathcal{V}$, have to be dynamically indistinguishable.

**IV. C-RELATED CONTROL SYSTEMS**

In order to overcome limitations [5] imposed on the choice of the aggregation matrix $C$ by the invariance requirement of $\text{Ker } C$, a generalization of system aggregation, called model abstraction, has been proposed in [10], extending the notion of $\Phi$-related vector fields to $\Phi$-related control systems (only for $\Phi = C$ epic).

**Definition 4.1:** (C-related control systems)

Consider the linear control systems

$$(\Sigma_o) \quad \dot{x} = Ax + Bu \quad x \in \mathcal{X} \quad u \in \mathcal{U}$$

$$(\Sigma_o) \quad \dot{y} = Fy + Gu \quad y \in \mathcal{V} \quad v \in \mathcal{W}$$

and a surjective map $y = Cx$. Control systems $\Sigma_o$ and $\Sigma_r$ are $C$-related if and only if $\forall x \in \mathcal{X}$ and $\forall u \in \mathcal{U}$ there always exists $v \in \mathcal{W}$ such that

$$C(Ax + Bu) = FCx + Gu \tag{7}$$

Then $\mathcal{V}_o$ is said to be an abstraction of $\mathcal{V}_r$ and generates trajectories of the form $y(t) = Cx(t)$, where $x(t)$ is a trajectory of $\mathcal{V}_r$.

The main difference between aggregation and abstraction is that abstraction does not necessarily have the same inputs as the original system; not only it may preserve all or some or even none of them, but it may also present additional ones.

In [10] it is shown that, given a system $\Sigma_o$ and a surjective map $y = Cx$, an abstraction $\Sigma_o$ of $\Sigma_o$ always exists. In particular the following proposition gives a canonical construction to generate abstractions.

**Proposition 4.2:** Consider the linear system

$$(\Sigma_o) \quad \dot{x} = Ax + Bu$$

and a surjective map $y = Cx$. Let

$$(\Sigma_o) \quad \dot{y} = Fy + Gu$$

be the system where

$$F = CA^+ \quad G = \left[ CB \quad CAS_1 \ldots \quad CAS_r \right]$$

with $C^+$ the Moore-Penrose pseudoinverse of $C$ and $s_1, \ldots, s_r$ span $\text{Ker } C$. Then $\Sigma_o$ is $C$-related to $\Sigma_r$.

The above abstraction $\Sigma_o$ is defined as the canonical abstraction. Relation (2) between trajectories stated in Definition 4.1 implies controllability (stabilizability) propagation from the original $n$-dimensional system to its $m$-dimensional abstraction

$$C \mathcal{R}(A, B) \subseteq \mathcal{R}(F, G) \quad (C \mathcal{S}(A, B) \subseteq \mathcal{S}(F, G))$$

where $\mathcal{R}(A, B) = \text{Im } \left[ B \quad AB \ldots \quad A^{n-1}B \right]$ and $\mathcal{S}(A, B) = \mathcal{X}^- + \mathcal{R}(A, B)$ are respectively the reachable subspace from the origin and the stabilizable subspace of $\Sigma_o$ and $\mathcal{X}^-$ is the stable subspace. In particular if $\Sigma_o$ is controllable (stabilizable) then $\Sigma_o$ is too. From the hierarchical point of view the reverse propagation is more interesting: the main result in [10] ([11]) states conditions under which controllability (stabilizability) equivalence of a system and its abstraction holds, i.e. controllability (stabilizability) also propagates from the abstraction to the original system. For the canonical abstractions the aforementioned conditions are fairly simple to check, as highlighted in the following Theorem from [10].

**Theorem 4.3:** Consider the linear system

$$(\Sigma_o) \quad \dot{x} = Ax + Bu$$

and a surjective map $y = Cx$. Let $\Sigma_o$ be the canonical abstraction and assume that

$$\text{Ker } C \subseteq \mathcal{R}(A, B) \quad (\text{Ker } C \subseteq \mathcal{S}(A, B))$$

Then $\Sigma_o$ is controllable (stabilizable) if and only if $\Sigma_o$ is controllable (stabilizable).
A. On the choice of the abstraction map

If the original system $\Sigma_o$ is controllable, i.e. $\mathcal{R}(A, B) = \mathcal{X}$, then conditions (8) are trivially satisfied for each possible map $y = Cx$. The choice of the abstraction map $C$ should be essentially driven by considerations on the process, for example time scale separation, economic or safety characteristics, etc.. We will highlight here two special cases which represent opposite situations.

Aggregation If $\mathcal{V} = \text{Ker} \ C$ is an $A$-invariant subspace, then $F = CAC^+ \Sigma$ is the quotient of $A$ w.r.t. map $y = Cx$, i.e. the following condition holds

$$FC = CA$$

Moreover for each $s_i \in \text{Ker} \ C$ one has $A_{si} \in \text{Ker} \ C$ hence $CA_{si} = 0$, and therefore

$$G = CB$$

which are the well-known conditions for $\Sigma_{\alpha}$ to be an aggregation of $\Sigma_o$. Note that the inputs of the abstracted system are those of the original system, i.e. $v = u$.

Pure abstraction The opposite situation arises when $\text{Ker} \ C \supseteq \text{Im} \ B$ (thus $CB = 0$), i.e. only virtual inputs appear in the abstracted system. These abstractions have been used in [10] to define controllability test algorithms. In particular, if $\text{Ker} \ C = \text{Im} \ B$ then all the states directly reached by the inputs are ignored in the abstraction and appear as the only inputs of the abstracted system. We will refer to this case as pure abstraction. Definition of virtual inputs in backstepping design follows this type of abstraction.

V. IMPLEMENTATION: PROBLEM DEFINITION

Now we have to define what we would like to obtain after an abstraction has been generated. For example one could be interested in designing a control law at the higher hierarchical level, thus solving a smaller size problem and obtaining the desired behavior of the variables of interest, and then implement this control law on the full-order model. We can state the problem in the following way.

Problem 5.1: (Control Implementation)
Consider the linear system

$$(\Sigma_o) \quad \dot{x} = Ax + Bu$$

and its canonical abstraction w.r.t. surjective map $y = Cx$

$$(\Sigma_{\alpha}) \quad \dot{y} = Fy + Gv$$

Suppose that the control law $v = K_u y$ is such that the closed-loop abstraction $\dot{y} = \Sigma_{\alpha}^C y$, with $\Sigma_{\alpha}^C = F + GK_u$, meets specifications for $y(t)$. Define a low-level control law $u = K_u x$, such that, for the closed-loop original system $\dot{x} = \Sigma x$, with $\Sigma = A + B K_u$, the following holds

i) variables of interest $y = Cx$ behave similarly to the closed-loop abstraction;

ii) states ignored in abstraction, i.e. those in Ker $C$, are accomodated in some desired way (e.g. stabilized), without affecting the performance of $y$.

Relation (2) allows to reformulate requirements i) and ii) in terms of trajectories of $\Phi$-related vector fields with two different approaches.

$C$-related approach Suppose there exists a control law $u = K_{\alpha}^c x$ such that $\mathcal{V} = \text{Ker} \ C$ is $\Sigma_{\alpha}$-invariant, with $\Sigma_{\alpha} = \Sigma + BK_{\alpha}^c$, and moreover

$$C \Sigma_{\alpha} = \Sigma_{\alpha}^C \Sigma \quad \iff \quad \Sigma_\alpha |_{\mathcal{V}} = \Sigma_{\alpha}^C$$

(9)

$$\Sigma_{\alpha} \Pi_c = \Pi_c Q \quad \iff \quad \Sigma_{\alpha} |_{\mathcal{V}} = Q$$

(10)

where $\text{Im} \ \Pi_c = \text{Ker} \ C$ and $\sigma(Q) \in \mathcal{C}^-$.

- Condition (9) states that $\Sigma_{\alpha}$ and $\Sigma_{\alpha}^C$ are $C$-related, and thus, using (2)

$$\dot{y}(t, Cx_0) = Cx(t, x_0) \quad \forall t$$

(11)

where $y(t, y_0)$ is the solution of $\dot{y} = \Sigma_{\alpha}^c y$ starting at $y_0 = Cx_0$.

- Condition (10) states that $\Sigma_{\alpha}$ and $Q$ are $\Pi_c$-related, and thus, if $x_0 = \Pi_c y_0$ i.e. if $x_0 \in \mathcal{V}$

$$x(t, \Pi_c z_0) = \Pi_c z(t, z_0) \quad \forall t$$

(12)

where $z(t, z_0)$ is the solution of $\dot{z} = Q z$ starting at $z_0 = \Pi_c^+ x_0$.

If $x_0 \notin \text{Im} \ \Pi_c$, provided $\Sigma_{\alpha}^c$ is asymptotically stable,

$$x(t, x_0) \rightarrow \Pi_c z(t, z_0) \quad \text{as} \quad t \rightarrow \infty$$

where $z_0 = (\Pi_c^+ + H_z) x_0$, with $H_z$ computable depending, among others, on $Q$. Note that, if $\Sigma_{\alpha}^c$ is only stable, the obtainable desired behavior of the variables of interest is determined by $\Sigma_{\alpha}^c$ and $y_0 = Cx_0$, as illustrated in Fig. 3, where $\Sigma_{\alpha}^c$ has pure imaginary eigenvalues.

For a generic initial condition $x_0$, the closed-loop system trajectories $x(t, x_0)$ will stabilize on a $\Sigma_{\alpha}$-invariant subspace, while $y(t)$ evolves according to $\dot{y} = \Sigma_{\alpha}^c y$ with $y_0 = Cx_0$. For a different initial condition $x_0$ trajectories will stabilize on another trajectory of the same

$1$Trajectories of $\hat{y} = \Sigma_{\alpha}^c y$ are on the quotient space $\mathcal{X}/\mathcal{V}$ which is not a subspace of $\mathcal{X}$; anyway we can represent $y$ on $\mathcal{V}$ which is isomorphic to $\mathcal{X}/\mathcal{V}$.

Fig. 3. Closed-loop trajectories with $C$-related approach
invariant subspace, with \( y(t) \) evolving like the solution of \( \dot{y} = \Sigma_\alpha y \) starting at \( y_0 = Cx_0 \). On the other hand, if \( x_0 \in \text{Im} \ C \), as in \( x_0 = Cx_0 \), then \( \dot{y}(t, x_{0}) \) will remain constrained in \( \text{Ker} C \) and approach the origin with the dynamics of \( Q \).

**II-related approach** Suppose there exists a subspace \( \mathcal{V}_\Pi \) and a control law \( u = K_u x \) such that

\[
\Sigma_{\Pi} x = \Sigma_{\Pi}^d x, \quad \Sigma_{\Pi} y = \Sigma_{\Pi}^d y \quad (13)
\]

\[
C_{\Pi} x = Q C_{\Pi} \quad \Sigma_{\Pi} x = \Sigma_{\Pi}^d x \quad (14)
\]

where \( \Sigma_{\Pi} = A + BK_u \) and \( \text{Ker} C_\Pi = \text{Im} \Pi = \mathcal{V}_\Pi \).

- Condition (13) states that \( \Sigma_{\Pi} \) and \( \Sigma_{\Pi}^d \) are \( C \)-related, and thus, if \( x_0 = \Pi y_0 \), i.e. if \( x_0 \in \mathcal{V}_\Pi \),

\[
x(t, \Pi y) = \Pi y(t, y_0) \quad \forall t
\]

where \( y(t, y_0) \) is the solution of \( \dot{y} = \Sigma_\alpha y \) starting at \( y_0 = \Pi^+ x_0 \).

- Condition (14) states that \( \Sigma_{\Pi} \) and \( Q \) are \( C \)-related, and thus

\[
z(t, C_\Pi x) = \Sigma_{\Pi}^d x(t, x_0) \quad \forall t
\]

where \( z(t, C_\Pi x) \) is the solution of \( \dot{z} = Qz \) starting at \( z_0 = C_\Pi x_0 \).

If \( x_0 \notin \text{Im} \Pi \), provided \( Q \) is asymptotically stable,

\[
x(t, x_0) \to \Pi y(t, y_0) \quad \text{as} \quad t \to \infty
\]

where \( y_0 = (\Pi^+ + H_y) x_0 \), with \( H_y \) computable, depending among others, on \( Q \). In terms of trajectories of the variables of interest \( y = Cx \) we can see that (Fig. 4)

- if \( x_0 \in \mathcal{V}_\Pi \), as in \( x_0^d \), then

\[
y(t, C^d x) = \Sigma_{\Pi}^d C \Pi y(t, y_0)
\]

with \( y_0 = \Pi^+ x_0^d \)

- if \( x_0 \notin \mathcal{V}_\Pi \), then

\[
y(t, C x_0) \to \Pi y(t, y_0)
\]

with \( y_0 = (\Pi^+ + H_y) x_0 \).

**Remark 5.2**: Note that with the \( C \)-related approach, (11) ensures that the variables of interest evolve as in the closed-loop abstraction for all \( t \), while with II-related approach there is only a similitude relation verified for any \( t \) (17) or asymptotically (18). On the other hand, in order to apply \( C \)-related approach, particular conditions must hold: Ker \( C \) has to be \( A \)-invariant (or \( (A, B) \)-invariant) and modes in \( A | V \) must be stable or stabilizable without affecting invariance of \( \mathcal{V} \). In the following subsection we will show some results in these special cases, while in the next section the general case will be treated with the II-related approach.

\[\text{Fig. 4. Closed-loop trajectories with II-related approach}\]

**A. Case Ker \( C \) \( A \)-invariant**

As stated in section IV-A if \( \mathcal{V} = \text{Ker} C \) is \( A \)-invariant, then abstraction \( \Sigma_\alpha \) reduces to an aggregation and assumes the form

\[
(\Sigma_u) \quad \dot{y} = CAC^+ y + CB u
\]

Then a direct expansion of the control law \( u = K_v y \) to control law \( u = K_v C x \) yields a closed-loop system \( \Sigma_c = (A + BK_c C) \) such that

\[
C(A + BK_c C) = (F + GK_c) C \quad \Sigma_c |_{X/V} = \Sigma^d_c
\]

\[
(A + BK_c C) \Pi_c = A \Pi_c - \Pi_c A |_{V} \quad \Sigma_c |_{V} = A |_{V}
\]

where \( \Pi_c = \text{Ker} C = \mathcal{V} \). The above relations state that invariance of \( \mathcal{V} \) has been preserved and that \( \Sigma_c \) and \( \Sigma^d_c \) are \( C \)-related, hence condition (9) is satisfied; partial state feedback \( u = K_c C x \) modifies the eigenvalues of \( A |_{X/V} \) without affecting those in \( A |_{V} \). In geometric terms \( \mathcal{V} \) has been rendered externally stable, but its internal stability properties have not been modified. If \( A |_{V} \) is stable, then requirement ii) of Problem 5.1 is satisfied. In [5] several methodologies have been proposed to stabilize modes in \( A |_{V} \). We briefly recall main results and show connections with present work.

1) Suppose \( \mathcal{V} \) is complementable\(^3\) w.r.t. \( \Sigma_c \) with complement \( T \); then we can perform a new aggregation on \( X/T \) with canonical projection map \( y_T = C_T x \), satisfying Ker \( C_T = T \). The new aggregated system is

\[
\dot{y}_T = F_T y_T + G_T u
\]

We can then choose a control law \( u = L_T y_T \) such that \( (F_T + G_T L_T) = Q \) for a suitable \( Q \). The closed-loop

\[\text{\(3\) is said to be complementable w.r.t. \( \Sigma_c \) if there exists a \( \Sigma_c \)-invariant subspace \( T \subset X \) such that \( V \oplus T = X \); then \( \mathcal{V} \) and \( T \) decompose \( X \) relative to \( \Sigma_c \), and there exists a coordinate change \( \dot{z} = T x \) such that}

\[
\dot{\Sigma}_c = T \Sigma^d_c T^{-1} = \begin{bmatrix} \Sigma^d_c |_{S} & 0 \\ 0 & \Sigma^d_c |_{T} \end{bmatrix}
\]

\[\text{with}
\]

\[
\Sigma^d_c |_{S} = \Sigma^d_c |_{X/S}
\]

\[
\Sigma^d_c |_{T} = \Sigma^d_c |_{X/T}
\]

\[\text{where}
\]

\[
\Sigma^d_c |_{S} = \Sigma^d_c |_{X/S}
\]

\[
\Sigma^d_c |_{T} = \Sigma^d_c |_{X/T}
\]
system $\dot{x} = \Sigma_T x$, with $\Sigma_T = (\Sigma_c + BL_T C_T)$, is then such that
\[
\begin{align*}
\Sigma_T |_{X/T} &= (F_T + G_T L_T) = Q \\
\Sigma_T |_{\Sigma_T} &= \Sigma |_{T}
\end{align*}
\]
Now $T$ is $\Sigma_T$-invariant, while $V$ is not. Moreover $\Sigma_{X/T}$ is stable by construction, while $\Sigma |_{T}$ is similar to $\Sigma_{X/T}$ which is stable after the first aggregation. Since invariance of $\text{Ker} C$ is not preserved, this solution can be casted in the $\Pi$-related approach with $\text{Im} \Pi = T$ rather than the $C$-related one.

b) Trying to directly impose condition (10), stabilization of the closed-loop system restriction to $V$ could be achieved if $V \subset \text{Im} B$ which clearly cannot be satisfied for single-input systems.

B. Case $K$ (A, B)-invariant

If $\bar{u} = \Gamma x$ is the control law which renders $K$ invariant, then the previous section results can be applied to system $\dot{x} = \bar{A} x + B \bar{u}$, where $\bar{A} = A + B \Gamma$.

VI. IMPLEMENTATION: $\Pi$-RELATED APPROACH

When conditions for $C$-related approach do not hold, we can apply the $\Pi$-related approach. A useful tool to treat this case is Immersion & Invariance; we briefly recall main result in [3] adapted to linear systems.

**Theorem 6.1:** Consider the linear system
\[
\dot{x} = Ax + Bu \quad x \in \mathcal{X} \quad u \in \mathcal{U}
\]
and the asymptotically stable system
\[
\dot{w} = Sw \quad w \in \mathcal{W}
\]
and suppose there exist matrices
\[
\Pi \in \mathbb{R}^{n \times m}, \quad \Gamma \in \mathbb{R}^{m \times n}, \quad K = [K_x \ K_z] \in \mathbb{R}^{k \times (n + m)}
\]
where $n$, $m$ and $k$ are the dimensions of $\mathcal{X}$, $\mathcal{W}$ and $\mathcal{U}$ respectively, such that the following conditions hold

a. Immersion & Invariance condition
\[
(A + B \Gamma') \Pi = \Pi S
\]
b. System
\[
\begin{align*}
\dot{z} &= \Phi (Ax + BK_x x + BK_z z) \\
\dot{x} &= Ax + BK_x x + BK_z z
\end{align*}
\]
where $\Phi = \text{Im} \Pi$, is asymptotically stable.

Then the closed-loop system $\dot{x} = (A + B K_x x + BK_z z) x$ is asymptotically stable. To understand how this results copes with the requirements of Problem 5.1, observe that

i) Condition (22) states that there exists a feedback\(^4\) control law $\bar{u} = \Gamma x$ that renders invariant a certain subspace $\mathcal{V}$ spanned by the columns of $\Pi$. Moreover $S$ is the restriction of $(A + B \Gamma')$ to $\mathcal{V}$. Stability of $S$ is equivalent to internal stability of $\mathcal{V}$.

ii) Matrix $\Phi$ is an implicit description of $\mathcal{V}$ since $\mathcal{V} = \{ x \in \mathcal{X} | \Phi x = 0 \}$. The quantity $z = \Phi x$ describes then how “far” we are from $\mathcal{V}$: for this reason it is referred to as off-manifold coordinate. Condition b. requires the existence of a control law $u = (K_x + K_z \Phi) x$ which renders $\mathcal{V}$ attractive (i.e. externally stable). Then off-manifold dynamics converge to zero and system (20) will asymptotically behave like the target system (21).

I & I provides a hierarchical stabilization procedure which allows to split a large scale control problem in two coarser problems

- computation of the subspace $\mathcal{V}$
- off-manifold stabilization.

We will now present off-manifold stabilization from a different perspective. Suppose $\bar{u} = \Gamma' x$ is the feedback which renders $\mathcal{V} = \text{Im} \Pi$ invariant and internally stable. In order to obtain its external stabilization we can perform an aggregation\(^5\) w.r.t. map $z = \Phi x$

\[
F = \Phi (A + B \Gamma') \Phi^+ \quad G = \Phi B
\]
Let $K_f$ be a feedback such that $(F + G K_f)$ is Hurwitz. As noted in section V-A feedback $\bar{u} = K_f \Phi x$ stabilizes externally $\mathcal{V}$ without affecting its internal stability. The overall control law $u$ will be $u = \bar{u} + \bar{u}$ i.e.

\[
u = (\Gamma' + K_f \Phi) x
\]
which emphasizes contributions of each design step. For general multi-input systems $K_x$ and $K_z$ are unique.

We can now use the above notions to solve problem 5.1 with the $\Pi$-related approach.

**Proposition 6.2:** Consider the linear system

\[
(\Sigma_o) \quad \dot{x} = Ax + Bu
\]
and its canonical abstraction w.r.t. surjective map $y = C x$

\[
(\Sigma_o) \quad \dot{y} = F y + G v
\]
Let $v = K_{x,y}$ be a control law such that the closed-loop abstraction $\dot{y} = \Sigma_o \dot{x}$, with $\Sigma_o = (F + G K_f)$, meets the desired behavior for $y$. Then there exists a control law $u = K_{u} x$ which implements the control law $v = K_{x,y}$ on $\Sigma_o$ in the sense of $\Pi$-related approach, and is derived as follow:

- find a control law $\bar{u} = \Gamma' x$ and a subspace $V_{\Pi} \subset \mathcal{X}$ such that
  \[
  (A + B \Gamma') \Pi = \Pi S_{\alpha}
  \]
  where $\text{Im} \Pi = V_{\Pi}$
- find a control law $\bar{u} = K_f \bar{z}$ which assigns the desired dynamics $Q$ to the aggregated system
  \[
  \dot{z} = F_z z + G_z \bar{u} \quad \text{with} \quad F_z = \Phi (A + B \Gamma') \Phi^+ \quad G_z = \Phi B
  \]
i.e. $(F_z + G_z K_f) = Q$ with $\text{Ker} \Phi = V_{\Pi}$.

Then $K_u = \bar{u} + \bar{u} = (\Gamma' + K_f \Phi) x$.

**Remark 6.3:** Immersion condition (22) admits a solution for any target system $\mathcal{S}$. Note however that, if the target system is a closed-loop abstraction, i.e. $S = \Sigma_o$, then relations (17)-(18) ensure in addition that the variables

\(^4\)Note that this is not a feedforward control.

\(^5\)Aggregation is possible since $\mathcal{V}$ is $(A + B \Gamma')$-invariant.
of interest \( y = Cx \) evolve in some a-priori known well-defined way. For the particular case of pure abstraction \( \Pi \), \( \Gamma' \) and \( K_u \) can be easily found in closed-form as shown next.

A. Control implementation in pure abstraction

II-related approach has in this case a very simple solution for \( \Pi, \Gamma' \) and \( K_u \) and a clear geometric interpretation. As pointed out in section IV-A if

\[
\ker C = \text{Im} \ B
\]

then the abstraction is an \( n-k \) dimensional systems, with the same number of inputs as the original system

\[
\dot{y} = CAC^+ + CABv \quad y \in \mathcal{Y} \quad v \in \mathcal{W}
\]

with \( \dim(\mathcal{Y}) = n-k \) and \( \dim(\mathcal{W}) = k \). States ignored by the abstraction map \( C \) enter as the only inputs of the abstracted system. In other words control law \( v = K_yy \) represents itself a specification for the evolution of the ignored variables in order for the variables of interest to behave as desired. We will show in the next Theorem how this simplifies control synthesis.

**Theorem 6.4:** Consider the linear system

\[
\dot{x} = Ax + Bu \quad x \in \mathcal{X} \quad u \in \mathcal{U}
\]

and the abstraction map \( y = Cx \) such that

\[
\ker C = \text{Im} \ B
\]

Choose a control law that stabilizes the abstracted system with desired eigenvalues

\[
v = K_y y
\]

and denote by \( \Sigma_{cl}^\alpha \) the closed-loop abstracted system

\[
\dot{y} = \Sigma_{cl}^\alpha y \quad \text{with} \quad \Sigma_{cl}^\alpha = CAC^+ + CABK_v
\]

Then

1) *Immersion condition* \( (A + B\Gamma') \Pi = \Pi \Sigma_{cl}^\alpha \) is satisfied by the insertion map

\[
\Pi = C^+ + BK_v
\]

where \( \ker \Phi = \mathcal{Y} = \text{Im} \Pi \) and the feedback

\[
\bar{u} = \Gamma' x = \Phi A x
\]

2) *External stabilization of \( \mathcal{Y} \) is achieved with the control law*

\[
\bar{u} = K_f \Phi x = -Q \Phi x
\]

where \( Q \) are the desired closed-loop off-manifold dynamics.

**Proof** First note that with the given II immersion condition becomes

\[
(A + B\Gamma') \Pi = \Pi C A \Pi \rightarrow B\Gamma' \Pi = (\Pi C - I) \Pi A
\]

which is satisfied by \( \Gamma' = \Phi A \) being

\[
(P\Pi C - I) - B\Phi F = C^+ C + BK_v C - I - BK_v C + BB^+ = 0
\]

since \( C^+ C = I - \ker C \ker C^+ \).

External stabilization of \( V \) is achieved with the aggregation map \( z = \Phi x \) operating on \( (A + B\Gamma') \)

\[
F = \Phi (A + B\Phi A) \Phi^+ \quad G = \Phi B
\]

Since \( \Phi B = (K_v C - B^+) B = -I \), then

\[
F = 0 \quad G = -I
\]

and \( K_f = -Q \) assigns the desired off-manifold dynamics

\[\hat{z} = Q z.\]

\[\Box\]

VII. CONCLUSIONS AND FUTURE WORK

In this paper we have addressed the problem on how a high-level control law, designed for an abstracted system, is implemented on the low-level original system. We focused on the hierarchical consequences of the abstraction map choice and have given explicit solutions for the pure aggregation case. Interpretation of the results in a geometric framework allows a deeper understanding of hierarchical control.

Future work will address the problems of finding closed-form solutions for the general case and of understanding the consequences of more specific requirements, e.g. tracking, on the variables of interest. Similar considerations could be done for the ignored variables. Further investigations will seek possible extensions of these ideas to the nonlinear case.

**REFERENCES**


