Algebraic Control of Unstable Delayed First Order Systems Using RQ-meromorphic Functions

L. Pekař, R. Prokop and R. Matusů

Tomas Bata University in Zlín, Faculty of Applied Informatics
Nad Stráněmi 4511, 760 05 Zlín, Czech Republic
E-mail: Pekar@fai.utb.cz, Prokop@fai.utb.cz

Abstract-The paper is focused on control of first order unstable delayed systems. The control design is performed in the $R_{MS}$ ring of retarded quasipolynomial (RQ) meromorphic functions. Unstable systems are modeled in anisochronic philosophy as a ratio of quasipolynomials where also denominator contains delay terms. The goal is to find a suitable stable quasipolynomial as a common denominator of $R_{MS}$ terms. This task is equivalent to the stabilization of a plant by a proportional controller in a feedback loop. Then, the appropriate controller can be found. In this paper, an algebraic method based on the solution of the Bézout equation with Youla-Kučera parameterization is presented. Besides the simple feedback loop, significant improvement using two-degrees of freedom structure is demonstrated. The method offers a real positive real parameter $m_0$ which defines closed loop poles placement. The modified “equalization method” for determining of $m_0$ can be applied. An example illustrates the proposed methodology, properties and benchmarking of all principles.

Keywords: Stabilization, delayed systems, algebraic control design.

I. INTRODUCTION

Algebraic tools play ever more significant role in modern control theory. Unlike some traditional state-space models, algebraic approaches are based on fractional description of systems. Any transfer function can be expressed as a ratio of two elements in the appropriate ring. Traditional transfer function is represented as a polynomial fraction. Another frequently used ring, besides polynomials, is designed as $R_{PS}$, Hurwitz stable a proper rational fractions. However, these fractional representations are heavily restrictive for anisochronic systems with delays. An elegant description of delayed systems, which combines differential and delay operators, is constituted by the ring of RQ-meromorphic functions, $R_{MS}$, [1–2]. An element of the ring can be described as a ratio of two quasipolynomials $y(s)/x(s)$. A quasipolynomial $x(s)$ of degree $n$ means

$$x(s) = s^n + \sum_{i=0}^{n-1} \sum_{j=1}^{\delta} x_{ij} s^i \exp(-\theta_{ij} s)$$  (1)

Quasipolynomial (1) is stable iff it owns no finite zero $s_0$ such that $\Re \{s_0\} \geq 0$. Stability can be verified by the Mikhailov stability criterion [7]:

$$\lim_{\omega \to \omega_0} \arg \{x(s)\} = (j\omega)^n + \sum_{\tau=0}^{n-1} \sum_{k=1}^{\delta} x_{\tau k} (j\omega)^k \exp(-\tau_{\tau k} j\omega) = n \frac{\pi}{2}$$  (2)

The numerator $y(s)$ of an element in $R_{MS}$ can be factorized in the form $y(s) = \tilde{y}(s) \exp(-\tau \omega)$, where $\tau > 0$ and $\tilde{y}(s)$ is a retarded quasipolynomial of degree $l$. 

II. DESCRIPTION OF DELAYED SYSTEMS USING MEROMORPHIC FUNCTIONS

Traditional transfer functions as a ratio of two polynomials are not suitable for models containing delays due to exponentials resulting from delays. Delayed systems can be naturally performed in the ring of stable and proper RQ-meromorphic functions, $R_{MS}$.

Any function over this ring is a ratio of two quasipolynomials $y(s)/x(s)$. A quasipolynomial $x(s)$ of degree $n$ means
The quasipolynomial fraction is called proper iff \( l \leq n \).

As a special case, a first-order unstable delayed plant is described by the transfer function

\[
G(s) = \frac{B(s)}{A(s)} = \frac{K \exp(-\tau s)}{Ts - \exp(-\tau s) + r_K \exp(-\tau s)}
\]

where \( A(s), B(s) \in \mathbb{R}^n \).

A class of models of form (4) represents so-called anisochronic systems [7 - 8] and these models can approximate dynamics of systems of very high order in the Laplace. These models have transcendental character, i.e. infinite spectrum. In some cases \( m(s) \) can be a polynomial; this choice is suitable for stable systems due to its simplicity. The necessity of the quasipolynomial form of \( m(s) \) rather than polynomial one for unstable delayed systems follows from the fact that a polynomial denominator \( m(s) \) can cause leaving of the RMS ring [9].

This fact can be easily illustrated by any example utilizing below control design methodologies.

III. STABILITY CONDITIONS

The key point of factorization (4) is to select a stable quasipolynomial \( m(s) \). Stability condition satisfying Mikhailov criterion (2) can be achieved by the appropriate choice of a real parameter \( r_0 \).

**Theorem 1:** Consider a quasipolynomial \( m(s) \) of the form

\[
m(s) = Ts - \exp(-\tau s) + r_K \exp(-\tau s)
\]

where \( K, T, \vartheta, \tau \) are non-negative real parameters. If the following condition for \( r_0 \) is valid

\[
1 < r_0 < \frac{T \omega_c + \sin(\vartheta \omega_c)}{K \sin(\tau \omega_c)}
\]

then \( m(s) \) is stable according to (2).

The term \( \omega_c \) means “the critical frequency”, i.e. a frequency when \( m(j \omega) \) crosses the critical point \([0,0]\), see Figure 1. The proof of the theorem can be performed in two different ways.

**Proof 1:** Since the highest \( s \)-power in (5) equals to 1, the condition of stability (according to Mikhailov criterion) of this quasipolynomial is satisfied if

\[
\lim_{\omega \to \infty} \arg[m(j \omega)]_{s=j \omega} = \frac{\pi}{2}
\]

Thus, the characteristics \( m(j \omega) \) has to pass through the first and second quadrants and the following conditions

\[
\text{Re}[m(j \omega)]_{s=j \omega} > 0; \quad \text{Im}[m(j \omega)]_{s=j \omega} \to \infty
\]

follows from Fig. 2. Moreover, from the figure is also clear that the characteristics must cross the imaginary axis firstly on the positive part of the axis. The crossover frequency is indicated as \( \omega_c \).

Real and imaginary parts of \( m(j \omega) \) can be rewritten

\[
\text{Re} : -\cos(\vartheta \omega) + r_K \cos(\tau \omega)
\]

\[
\text{Im} : T \omega + \sin(\vartheta \omega) - r_K \sin(\tau \omega)
\]

Taking the first condition in (8), \( \text{Re}[m(j \omega)]_{s=j \omega} > 0 \), while \( \text{Im}[m(j \omega)]_{s=j \omega} = 0 \), the following expression results from (9)

\[
r_0 > \frac{1}{K}
\]

Consider now that \( m(j \omega) \) has to cross positive part of imaginary axis first, i.e. \( \text{Im}[m(j \omega)]_{s=j \omega} > 0 \) while \( \text{Re}[m(j \omega)]_{s=j \omega} = 0 \). Assuming the critical case, on stability border (Figure 1)

\[
\text{Re}[m(j \omega)]_{s=j \omega} = 0; \quad \text{Im}[m(j \omega)]_{s=j \omega} = 0
\]

the following condition for critical frequency \( \omega_c \), with respect to (9), is obtained

\[
\omega_c = \frac{1}{T} \left[ \cos(\vartheta \omega_c) \tan(\tau \omega_c) - \sin(\vartheta \omega_c) \right]
\]

\[
= \frac{1 - \sin[(\vartheta - \tau) \omega_c]}{T \cos(\tau \omega_c)}
\]

and the critical parameter \( r_{0c} \) reads

\[
r_{0c} = \frac{1}{K} \frac{T \omega_c + \sin(\vartheta \omega_c)}{\sin(\tau \omega_c)}, \omega_c \neq 0
\]
Since $\Im\{m(j\omega)\}_{\omega=0} > 0$ must be satisfied, equation (13) suggests the second margin for the stability interval for $r_0$, therefore

$$r_0 < \frac{1}{K} \frac{T\omega_c + \sin(\varphi\omega_c)}{\sin(\tau\omega_c)}; \omega_c \neq 0 \tag{14}$$

The second condition in (8), $\Im\{m(j\omega)\}_{\omega=\infty} \rightarrow \infty$, follows directly from (9).

Remarks: The calculation of the critical frequency $\omega_c$ is the crucial point for finding of the suitable interval for $r_0$ according to (6). This frequency results from the solution of nonlinear equation (12). Naturally, this equation gives an infinite number of solutions, nevertheless, the “smallest non-zero” one is taken in account. The trivial solution of (12), $\omega_c = 0$, provides a non-oscillating stability border, corresponding to (10). On the contrary, a general non-trivial solution yields an oscillating stability border, which agrees with (13). For numerical calculation of $\omega_c$, it is possible to take approximation (13) of sine and cosine functions according to McLaurin series (first two elements).

$$\sin(x) = x - \frac{x^3}{6}; \cos(x) = 1 - \frac{x^2}{2} \tag{15}$$

which yields the estimation for the critical frequency

$$\hat{\omega}_c = \sqrt{\frac{6(T + \tau - \vartheta)}{(\tau - \vartheta)^3 + 3T\tau}} \tag{16}$$

Equation (6) reveals the following condition

$$\frac{T\omega_c + \sin(\varphi\omega_c)}{\sin(\tau\omega_c)} > 1 \tag{17}$$

which guaranties non-empty interval for $r_0$.

Analogously, conditions for $r_0$ while $K < 0$ can be derived as

$$\frac{1}{K} > r_0 > \frac{1}{K} \frac{T\omega_c + \sin(\varphi\omega_c)}{\sin(\tau\omega_c)} \tag{18}$$

This contribution offers the possible range for $r_0$ so that $m(s)$ is stable. On the other hand e.g. [2], [10] suggests finding exact $r_0$ value by shifting the right-most pole of $m(s)$ into desired position using derivative of the characteristic equation.

Analog with proportional feedback stabilization

The above described searching of $r_0$ for stable quasipolynomial $m(s)$ is equivalent to stabilization of unstable system (4) with the aid of a serial proportional preliminary controller $r_0$ in the feedback loop, according to Fig. 3. The condition (6) can be obtained by Nyquist criterion and it gives a different way how to prove Theorem (1).

The equivalence arises from the transfer function of the closed loop $G(s)$

$$G_s(s) = \frac{Y_u(s)}{E_u(s)} = \frac{r_0K \exp(-\pi)}{T_s - \exp(-\pi) + r_0K \exp(-\pi)} \tag{19}$$

Hence, characteristic quasipolynomial of $G_s(s)$ equals to $m(s)$.

As “the principle of argument” is valid [7], Theorem (6) can be proved also using the Nyquist stability criterion.

Proof 2: Let the plant be modeled by the transfer function (4) and consider preliminary proportional controller $r_0$. Thus, the open loop transfer function containing $r_0$ and $G(s)$ is

$$G_o(s) = r_0 G(s) = \frac{r_0K \exp(-\pi)}{T_s - \exp(-\pi)} \tag{20}$$

In order to satisfy the Nyquist criterion, the Nyquist plot of (20) must circle round the critical point [-1, 0] only once, see Fig. 4. Hence, the following conditions must be valid

$$\Re\{G_o(j\omega)\}_{\omega=\omega_t} < -1 \quad -1 < \Re\{G_o(j\omega)\}_{\omega=\omega_t} < 0 \tag{21}$$

where $\omega_t$ is the crossover frequency, i.e. $\Im\{G_o(j\omega)\}_{\omega=\omega_t} = 0$.

Figure 3. Proportional closed loop
Now, let \( G_0(j\omega) \) be transcribed into real and imaginary parts as
\[
\begin{align*}
\text{Re} & : r_0\frac{K[\cos(t\omega)\cos(\vartheta \omega) - \sin(t\omega)(T_0 + \sin(\vartheta \omega))]}{(T_0)^2 + 2T_0\omega\sin(\vartheta \omega) + 1} \\
\text{Im} & : r_0\frac{K[\sin(t\omega)\cos(\vartheta \omega) - \cos(t\omega)(T_0 + \sin(\vartheta \omega))]}{(T_0)^2 + 2T_0\omega\sin(\vartheta \omega) + 1}
\end{align*}
\]  
(22)
and consider the first condition in (20), i.e. \( \text{Re}\{G_0(j\omega)\}_{t=0} < -1 \). Let us label
\[
\Delta(\omega) = \frac{-\cos(t\omega)\cos(\vartheta \omega) - \sin(t\omega)(T_0 + \sin(\vartheta \omega))}{(T_0)^2 + 2T_0\omega\sin(\vartheta \omega) + 1}
\]  
(23)
The initial condition \( \omega = 0 \) gives \( \Delta(0) = -1 \) and inequality (10), is directly obtained.

The second limitation for \( r_0 \) stems from the further relation in (21), i.e. \(-1 < \text{Re}\{G_0(j\omega)\}_{t=0} \). Presuming \( \Delta(\omega_1) > 0 \) firstly, \( r_0 > -\left((K\Delta(\omega))^{-1}\right) \) is obtained; which gives no limitation for \( r_0 \), since (10) has already been established.

By contrast, if \( \Delta(\omega_1) < 0 \), the following relation can be written
\[
r_0 < \frac{-1}{\Delta(\omega_1)K}
\]  
(24)

where the crossover frequency \( \omega_1 \) is given by (12), where indeed \( \omega_c = \omega_1 \). The desired second border in (6) is obtained by assuming (24) with respect to (12).

The last condition in (21) remains, i.e. \( \text{Re}\{G_0(j\omega)\}_{t=0} < 0 \). As \( K > 0 \), \( \Delta(\omega_1) < 0 \) must be satisfied; which has already been considered in (24). □

Remarks: It is also possible to formulate a tighter condition for \( r_0 \). Let the second relation in (21) be written in the following form
\[
-\frac{1}{A_m} < \text{Re}\{G_0(j\omega)\}_{t=0} < 0
\]  
(25)

Then \( A_m \) can be taken as a gain margin, thus, the final formula reads as
\[
\frac{1}{K} < r_0 < \frac{1}{A_mK}\frac{T_0\omega_c + \sin(\vartheta \omega_c)}{\sin(t\omega_c)}
\]  
(26)

IV. PARAMETERIZATION OF CONTROLLERS

A. One degree of freedom (1DOF) structure

The above time delay systems description now can be utilize for controller design. Let the control loop is of one degree of freedom (1DOF) structure, see Fig. 5. The plant transfer function is expressed by (4). Moreover, \( A(s) \) and \( B(s) \) are coprime (details about divisibility in \( R_M \) can be found in [1]) and the common denominator of \( A(s) \) and \( B(s) \) is taken with respect to (6).

The first step of the control design is to stabilize the system by a suitable feedback loop. If there exist functions \( Q_0(s), P_0(s) \in R_M \) which satisfy the Bézout identity
\[
A(s)P_0(s) + B(s)Q_0(s) = 1
\]  
(27)
then the set of all internally stabilizing controllers is given by parameterization form
\[
G_h(s) = \frac{Q_0(s) + A(s)Z(s)}{P_0(s) - B(s)Z(s)}, \quad P_0(s) - B(s)Z(s) \neq 0
\]  
(28)

where \( Z(s) \) is an arbitrary element in \( R_M \). The proof can be found in e. g. [1], [11]. The suitable choice of \( Z(s) \) can ensure additional control conditions. Consider reference, \( W(s) = H_M(s)/F_0(s) \), and load disturbance \( D_L(s) = H_M(s)/F_D(s) \), where \( H_M(s), F_0(s), H_M(s) \) and \( F_D(s) \) are over \( R_M \).

Conditions for asymptotic reference value tracking and disturbance attenuation result from expression for \( E(s) \) that reads
\[
E(s) = \frac{A(s)P(s)}{A(s)P(s) + B(s)Q(s)}W(s) - \frac{B(s)P(s)}{A(s)P(s) + B(s)Q(s)}D(s)
\]  
(29)

It is required \( E(s) \in R_M \), i.e., to reach \( E(s) = 0 \) for \( s \to 0 \). In other words, it is demanded that unstable factors of \( F_0(s) \) and \( F_D(s) \) must divide \( P(s) \). In the case that both signals \( w(t) \) and \( d(t) \) are step functions then the absolute term (coefficient for \( s^0 \)) in \( P(s) \) has to be equal to (30).

![Figure 4. Nyquist plot of a stable system](image)

![Figure 5. 1DOF control structure](image)
where \( m_0 \) is used as a “tuning knob.” Term (30) says that the controller \( G_m(s) \) has integral behavior. This condition is assured by proper choice of \( Z(s) \) in (28). If \( w(t) \) or \( d(t) \) are another functions, divisibility conditions can be more complex.

**B. Two degrees of freedom (2DOF) structure**

Control structure of the 1DOF type ensures stability, reference tracking and disturbance attenuation with aid of only one controller. Better control response can be obtained when reference tracking is decoupled from the stability fulfillment and disturbance attenuation. This is ensured by an additional controller \( G_C(s) = \frac{R(s)}{P(s)} \), see Fig. 6.

Stability and disturbance rejection is solved in the same way as for the 1DOF structure, i.e. with help of (27) and (28) with respect to the condition that \( F_D(s) \) divides \( P(s) \). Reference tracking condition arises from transfer function of reference \( W(s) \) to control error \( E(s) \)

\[
\frac{E(s)}{W(s)} = 1 - B(s)R(s)
\]  

(31)

with respect to (27). In order to \( E(s) \) be in \( R_{us} \), an unstable factor of \( F_m(s) \) must be obviated. If the reference is step function then the following condition thus have to be fulfilled

\[
\lim_{s\to\infty} \left[ 1 - B(s)R(s) \right] = 0
\]

(32)

This condition ensures at least one zero at \( s = 0 \) in (31) which agrees with the zero in \( F_m(s) \).

**V. DEMONSTRATIVE SIMULATION EXAMPLE**

Consider the model describing an unstable delayed process

\[
G(s) = \frac{3\exp(-4s)}{5s - \exp(-0.8s)}
\]

(33)

Solution of (12) gives critical frequency \( \omega_c = 0.238 \). However, estimation of \( \omega_c \) according to (16) is \( \omega_c = 0.728 \). The possible range of \( r_0 \) then results from (6) as interval \((0.333; 0.564)\). Let the gain margin is \( A_g = 1.3 \), which gives \( r_0 = 0.434 \).

Utilizing the 1DOF control structure, the Bézout identity (27) yields particular solution \( Q_0(s) = r_0 = 0.434 \), \( P_0(s) = 1 \). If \( Z(s) \) in the parameterization (28) is chosen as

\[
Z(s) = \frac{m_0m(s)}{K(s + m_0)} = \frac{m_0\left[1.3\exp(-4s) + 5s - \exp(-0.8s)\right]}{3(s + m_0)}
\]

then the controller has a structure

\[
G(s) = \frac{(1.3 + 5m_0)s + m_0\left[1.3 - \exp(-0.8s)\right]}{3s + m_0(1 - \exp(-4s))}
\]

(34)

The final controller (35) behaves as PI for \( s \to 0 \).

The controller tuning can be performed through a free parameter \( m_0 \). The question of the correct and optimal choice for \( m_0 \) has not been solved. For simulations example, which is pictured in Fig. 7, various random \( m_0 \) values were chosen.

If the 2DOF scheme is assumed, the feedback controller \( G_R(s) \) is calculated as (35) again. We recall that both \( w(t) \) and \( d(t) \) are step functions.

![Figure 6. 2DOF control structure](image)

![Figure 7. Simulation results for the 1DOF structure](image)
Reference tracking is ensured by the solution of the condition (32), in the concrete
\[
0 = \lim_{s \to 0} \left( 1 - \frac{3 \exp(-4s)}{1.3 \exp(-4s) + 5s - \exp(-0.8s)} \right) R(s)
\] (36)
The condition (36) is fulfilled if \( R(s) = r_0 - 1/K = 0.1 \). Then the feedforward controller \( G_c(s) \) reads
\[
G_c(s) = \frac{R(s)}{P(s)} = \frac{\left( r_0 - \frac{1}{K} \right)(s + m_0)}{(s + m_0) \exp(-\alpha s)} = \frac{0.1(s + m_0)}{(s + m_0) \exp(-4s)}
\] (37)
The simulation results for the 2DOF are displayed in Fig. 8. Values of tuning parameter \( m_0 \) are chosen for comparison as the same as for the 1DOF.

It is obvious from the figures that decoupling brings much better control responses. Tuning parameter \( m_0 \) has only minor influence over output signal. However, its increasing leads to faster changes of control signal, in case of the 1DOF.

CONCLUSION

The contribution brings a possible way of algebraic control approach for unstable first order delayed systems. A class of the systems is modeled in the anisochronic form containing delay terms also in a denominator of the transfer function. The key point of the design is to find a suitable stable quasipolynomial as a common denominator of terms over \( R_{\text{ag}} \). A methodology based on the solution of the Bézout equation with Youla-Kučera parameterization is described. Besides usual the one degree of freedom (1DOF) structure, the two-degrees of freedom (2DOF) control scheme is utilized. This improvement enables to decouple reference tracking from tasks of stabilization and disturbance rejection. The method offers a positive real parameter \( m_0 \). A simulation example showed a comparison of the proposed modifications.

ACKNOWLEDGEMENT

This work was supported in part by the grants of Ministry of Education of the Czech Republic MSM 708 835 2102.

REFERENCES