The Effect of Distance between Open-Loop Poles and Closed-Loop Poles on the Numerical Accuracy of Pole Assignment

E. Murat Bozkurt*, M. Turan Söylemez**, Member IEEE
Istanbul Technical University/Electrical Engineering Department, Istanbul, Turkey
* bozkurte@itu.edu.tr  ** soylemez@elk.itu.edu.tr

Abstract. The early works on the numerical performance of the pole assignment algorithms show that there are several parameters that affect the error made in calculations of the resulting compensators when fixed precision arithmetic is used. It has been already shown that increasing the order of the system and the ratio of real poles to all poles of the open-loop (and closed-loop) system (RPR) cause an exponential increase in the numerical errors made in pole assignment. In this paper, the effect of maximum distance between open-loop system poles and target closed-loop system poles (MD) on the performance of pole assignment algorithms is investigated for the first time.

Keywords: pole assignment, numerical accuracy, state feedback

I. INTRODUCTION

Linear control system theory is well advanced and several methods exist in order to achieve desired design objectives. Pole assignment is one of the most commonly used design techniques in control theory and a number of constructive methods in this area are described in the literature [1]. It has been long known that under certain conditions it is possible to assign the closed-loop system poles arbitrarily by state-feedback provided that the system is controllable [2]. From this point of view, the process of determining the required state feedback matrix \( K \) can be assigned to arbitrary, real or complex conjugate, locations if, and only if, \((A,B)\) is controllable [2]. The process of determining the required state feedback controller \( K \) usually involves numerical errors when the calculations are carried out in a numerical programming environment where a fixed precision arithmetic is used [3]. Actually, an exact pole assignment is possible when symbolic algebra is used. However, symbolic algebra is not always available and also it can be time consuming carrying out calculations exactly when it is available. Therefore error prone numerical calculations are sometimes unavoidable. Because of these numerical calculation errors, the resulting state feedback matrix of the pole assignment algorithm may be calculated defectively. Obviously, the use of a defectively calculated state feedback matrix may result in significant perturbations from the desired locations of closed-loop poles on the resulting closed-loop system especially for high order systems.

It is also a well-known fact that the accuracy of pole assignment plays a crucial role on both the robust stability and the performance of the resulting system. From this point of view, the determination of the parameters, which can be effective on the quantities of these perturbations, is necessary and helpful for robustness and performance of the resulting closed-loop system. To this end, the effect of the algorithm used and the degree of the system on the performance of pole assignment has been investigated in [3], and that of the ratio of real poles to all poles of the open-loop (and closed-loop) system (RPR) has been investigated in [4]. In this study, another important parameter, which is named maximum distance (MD) between open-loop system poles and target closed-loop system poles, effective on the performance of the pole assignment algorithms, has been investigated.

It is well known that there is no algebraic relation for the roots of polynomials of order greater than four in terms of their coefficients. Therefore, analyzing the numerical errors on the pole assignment algorithms by means of an algebraic method is difficult. For this reason, we shall follow a Monte Carlo approach in order to analyze the important parameters on the accuracy of the resulting compensator of the pole assignment algorithms.

On the other hand, the effect of both RPR of the open-loop system poles (ORPR) and also that of the closed-loop system poles (CRPR) on the performance of the pole assignment algorithms must be isolated when the effect of “MD” on the performance of pole assignment algorithms is investigated. Hence, in the following sections, first of
all, we shall explain briefly the notion of RPR in section II. We shall then introduce some special methods and tools in order to perform the required analysis about the effect of MD on the performance of the pole assignment algorithms in section III and IV respectively. The method used is explained in section V. This is followed by some considerations about the results of analysis in section VI. In the final section, conclusions and possible future research areas are given.

II. THE NOTIONS OF REAL POLE RATIO (RPR) and EXPECTED REAL POLE RATIO (ERPR)

Each possible parameter affecting the performance of the pole assignment algorithms must be isolated from others when one of them is to be analyzed. Therefore, the fundamental concepts of both RPR and ERPR and some simple methods in order to generate a random system with prescribed RPR will be explained briefly before providing the method of analysis.

Given a set of complex numbers (closed under complex conjugacy) \( \Gamma = \{ \gamma_i \mid \gamma_i \in \mathbb{C}, (i = 1, 2, \ldots, n) \} \) the Real Pole Ratio (RPR) is a real number that takes values between 0 and 1 and is defined as the ratio of the number of real elements to the total number of elements (\( n \)) in the set. That is,

\[
RPR(\Gamma) \triangleq \frac{\text{The number of real elements of } \Gamma}{n} \quad (3)
\]

Using this definition, it is possible to talk about the RPR of the open loop system poles (ORPR) and also that of the closed-loop system poles (CRPR). In any case, RPR takes discrete values for a given order of the system \( n \). If \( n = 5 \), for example, RPR can be \( \frac{1}{5}, \frac{3}{5} \) or 1 (note that complex roots of a polynomial with real coefficients appear in conjugate pairs). Similarly, for \( n = 6 \), RPR takes values from the set \( \{0, \frac{1}{3}, \frac{2}{3}, 1\} \). Although it could be meaningful to examine the effect of RPR to the performance of pole assignment at such discrete values for a given system order, it is difficult to make comparisons of the results of such an analysis for systems with different orders. Therefore the concept of expected value of RPR (ERPR) is used in the following analysis. ERPR is defined as the expected value of RPR for a randomly generated set of complex numbers. That is,

\[
ERPR(\Gamma) \triangleq \frac{\text{Expected value of number of real elements of } \Gamma}{n} \quad (4)
\]

Like RPR, it is possible to talk about ERPR of an open-loop system (EORPR), and that of a closed-loop system (ECRPR) for randomly generated open-loop systems and target sets of closed-loop system poles. It should be noted that if open-loop systems are generated from \( n \times n \) state matrices whose elements are independent random variables with standard distributions, EORPR becomes a function of \( n \) and decreases asymptotically as \( n \to \infty \) [5]. Therefore generating state-matrices directly from random elements is not enough to examine the effect of EORPR. Instead, a linear transformation from block diagonal to a prescribed random state-matrix is used for generating state-matrices in the following. In this approach, first a set of complex conjugate numbers \( \Gamma = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \) with a given ERPR is randomly generated as explained in [4]. Then, an open-loop system \((A, B)\) is generated such that

\[
A = T^{-1} \Lambda T \quad (5)
\]

where the matrices \( B \in \mathbb{R}^{n \times d} \) and \( T \in \mathbb{R}^{n \times n} \) are generated randomly, and \( \Lambda \in \mathbb{R}^{n \times n} \) is a block diagonal matrix as

\[
\Lambda = \begin{bmatrix}
\text{Re}[\lambda_1] & \text{Im}[\lambda_1] \\
\text{Im}[\lambda_1] & \text{Re}[\lambda_1] \\
\text{Re}[\lambda_2] & \text{Im}[\lambda_2] \\
\text{Im}[\lambda_2] & \text{Re}[\lambda_2] \\
\end{bmatrix}
\]

where \( \lambda_1 \) and \( \lambda_2 \) denote complex conjugate pairs.

III. POLE COLORING as an ERROR MEASURE for the ACCURACY of POLE ASSIGNMENT

It is a well known fact that the lack of accuracy on the resulting compensator of the pole assignment algorithms usually causes perturbations from the desired locations of the closed-loop poles. Let us suppose that the error measure is the maximum distance between the desired locations of closed-loop poles and the corresponding perturbed closed-loop poles. However, finding the error poses a new problem since it is not always easy to determine which of the desired poles correspond to which of actual closed-loop poles. Thus, we need to make a bijective (one to one) assignment between the desired locations of poles and the perturbed poles. This problem is referred as pole coloring [6].

The simple case of the desired locations of poles and corresponding perturbed poles for a third-order closed-loop system is given as in Figure 1.a. For the system shown in Fig.1.a there are six possible permutations. Three of them are shown in Figure 1.b, Figure 1.c, and Figure 1.d respectively. Let us define the maximum distance between paired desired and perturbed poles for \( p \)th permutation as "cost for \( p \)" and show it as \( J_p \) and the error for numerical pole assignment problem with \( J_p \).

Here, it would be reasonable to choose the permutation which has the smallest cost. Thus, fig.1.d is the proper choice from all possible permutations. Hence, the error for pole assignment can be written as

\[
J_p = \min_{i \neq p} \left( J_p \right)
\]

\[
= \min_{i \neq p} \max \left( d_i \right)
\]

where \( d_i \) shows the distance between the \( i \)th pair of assigned poles given by permutation \( p \). Solving this problem as defined by equation (4) could be very slow,
since there are \( n! \) permutation to test. Fortunately there are good algorithms to address this problem, which in fact is called the linear bottleneck assignment problem (BAP) in computer science terminology [7].

\[ \begin{align*}
\text{Fig. 1} & \quad \text{Desired closed-loop poles and perturbed closed-loop poles of a third order system is shown in (a) and possible permutations of assignment problem of this system are shown in (b), (c) and (d) respectively.}\\
\end{align*} \]

\( \Box \) Perturbed Poles \( \bigcirc \) Desired Poles

IV. DISTANCE BETWEEN OPEN-LOOP POLES and DESIRED CLOSED-LOOP POLES (MD)

In line with the error definition made above the maximum distance between open-loop system poles and target closed-loop system poles is called as maximum distance, and shortly denoted as MD.

It is possible to generate a target closed-loop characteristic polynomial where the maximum distance between its roots and open-loop pole locations is a given value. Another way of thinking, if the pole coloring procedure is applied between open loop poles of the given system and the newly generated set of complex numbers, the result is required to be a desired value of MD. Let us consider that a random test system with a given ERPR is produced randomly using the approaches provided in the previous sections. Then, in order to produce a set of complex numbers, closed under conjugacy with a given MD, a simple procedure with the following basic steps can then be used. Let us suppose that the desired value of MD is “\( \beta \)”. Then,

1. Select a pole (\( \lambda_i \)) randomly from the pole set of the open-loop system, for which ERPR is known.
2. If \( \lambda_i \) is real, then select a new point on the real axis \( \rho_i \) such that \( |\rho_i - \lambda_i| = \beta \). If \( \lambda_i \) is complex conjugate pair of \( \lambda_k \), then select a new \( \rho_i \) on the circle centered \( \lambda_k \) with radius \( \beta \) and its conjugate as \( \rho_2 \).
3. For all of remaining open-loop system poles \( \lambda_i (i = 2, \ldots, n) \), determine a target value \( \rho_i \) such that

\[ a) \rho_i \text{ is complex conjugate of } \rho_{i-1} \text{ if } \lambda_i \text{ is complex conjugate pair of } \lambda_{i-1} \]
\[ b) |\rho_{i-1} - \lambda_i| < \min \left( \left| \rho_{i-1} - \lambda_i \right|, \beta \right), \text{ otherwise (see Figure 2)} \]

\[ \begin{align*}
\text{Fig 2: The simple demonstration of the procedure which can be used in order to generate a target closed-loop characteristic polynomial with a certain MD.}\\
\end{align*} \]

V. THE METHOD

In this study, a Monte Carlo approach is pursued in order to examine the effect of MD on the performance of pole assignment algorithms. Here, a large number (usually 10000) open-loop systems with a certain OERPR and the corresponding target closed-loop system poles which has a certain MD form the open-loop system poles are produced randomly using the approaches provided in the previous sections. (Note that CERPR will be automatically same as OERPR when the approach given in the previous section is used.) Then, for each pair of open-loop system \((A,B)\) and target set of closed-loop system poles \( \gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \) a pole assignment algorithm is used to calculate the required state feedback matrix \((K)\). Generally, the spectral approach is used as the pole assignment method, since it is known to have better numerical properties in comparison to some of the other well known state feedback pole assignment methods for single input systems [3]. Due to spectral approach [1,8] the desired state-feedback matrix is given by

\[ K = \sum_{i=1}^{n} \zeta_i \nu_i^T \]  \hspace{1cm} (9)

where \( \nu_i^T \) are reciprocal eigenvectors (rows of the inverse eigenvector matrix), and \( \zeta_i \) are scalar weighting factors calculated as

\[ \zeta_i = \frac{\prod_{j=1}^{i} (\lambda_i - \gamma_j)}{\prod_{j=1}^{n} (\lambda_i - \lambda_j)} \] \hspace{1cm} (10)
where \( \rho_i \) are scalars given by
\[
\rho_i = \sqrt{i} v_i B
\]
(11)

After finding the state-feedback matrix, \( K \), the closed-loop system A matrix \( A_c = A - BK \) is calculated. The eigenvalues of this matrix gives the achieved set of closed-loop system poles, which usually are different than desired set of poles. Using the approach provided in section III, the distance between the achieved and the desired set is then calculated as the error of pole assignment. This calculation is carried out for each pair of open-loop system \((A, B)\), and target set of closed-loop system poles and the average (and also standard deviation) of the error is found for given value of MD.

VI. THE RESULTS OF THE ANALYSIS

In this section, we examine the effect of MD on the performance of the pole assignment. Therefore, in order to examine the effect of MD on the numerical accuracy of resulting compensator of the pole assignment, the expected value of both ORPR (OERPR) and CRPR (CERPR) are set to a fixed value (0.5), and the value of MD is changed between \(10^{-1}\) and \(10^7\). For each value of MD, 10000 test systems and a target set of closed-loop system poles for each system are generated. Then using the error analysis method given in the previous section, the average values of the errors and the standard deviations of the errors are found. The results are depicted in Figure 3. Note that the vertical axis in Figure 3 represents the logarithmic error (maximum distance) between the achieved and the desired sets of closed-loop system poles and horizontal axis represents the logarithm of the value of MD. It is possible to roughly state that it shows the accuracy of the achieved closed-loop system poles with respect to desired ones. For example, when MD is \(10^6\) the maximum distance between desired and achieved set of closed-loop systems is expected to be around \(10^{-7}\). It can be observed from the figure that the error made by pole assignment increases exponentially as MD changes between \(10^1\) and \(10^6\).

In addition, the effect of MD on the accuracy of the resulting compensator of pole assignment algorithm can be analyzed for different values of OERPR (and CERPR). The results for system orders 10 and 15 are depicted in Figure 4 and Figure 5, respectively. The relation between ERPR and MD can be examined more deeply when we regard the Figure 4.

The error made in pole assignment increases piecewise linearly as MD increases as it can be observed from Figure 4. The rate of the error curves, however, changes as ERPR changes between 0 and 1. This result indicates that the number of real valued poles can be considered as a nonlinear multiplier for the effect of MD on the errors made in pole assignment. A similar result is presented in Figure 5, where the system order is taken as 15. The difference between Figure 4 and Figure 5 is that the error curves are shifted upward in Figure 5, as expected.
Error curves for a fixed ERPR (0.5) are depicted on Figure 6 for different values of system orders (n=10, 15 and 20). For low value of MD, error curves are shifted upwards as the order of the system increases. An interesting fact observed in this figure, however, is that the value of Log(MD) is the dominating parameter in determination of the errors for high values of MD ($MD > 10^5$). This is quite surprising as one would expect the order of the system to be the main parameter in determination of the errors made in pole assignment.

Obviously, the algorithm used in pole assignment is an important factor that affects the errors on the achieved closed-loop system poles. It is a well known fact that the spectral method has better numerical properties in comparison to many of the other methods for state-feedback pole assignment. Nevertheless, examination of Figure 6 reveals that this statement is true only for smaller values of MD ($MD < 10^5$). For large values of MD, the performance of the spectral approach becomes worse in comparison to other methods. In particular, the error made in Ackermann’s method is observed to have smaller values when $MD > 10^5$ for systems with order 10 and RPR 0.5.

It should be noted that the numerical accuracy of pole assignment algorithms are also dependent on other parameters such as the condition number of the controllability matrix of a given system and the numerical accuracy of numerical calculations carried out. Therefore the results obtained here (numerical values of errors found through experiments) may change. However, we observe that the trends found in this study are consistent and does not change if all the remaining parameters are fixed.

Future research should include the effects of the other possible parameters such as the condition number of the controllability matrix of a given system to the performance of pole assignment algorithms. Also algebraic formulations to relate these parameters to the error should be sought.
REFERENCES


