An Impulsive Observer that Estimates the Exact State of a Linear Continuous-Time System in Predetermined Finite Time

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Abstract— This paper presents a new observer for the class of linear continuous-time systems. In contrast to many well-established observers, which normally estimate the system state in an asymptotic fashion, the proposed observer estimates the exact system state in predetermined finite time. The finite convergence time of the proposed observer is achieved by updating the observer state based on current observer data at a definite time instant. Simulation results are presented to illustrate the convergence behavior of the proposed observer.

I. INTRODUCTION

Many observer design techniques for linear continuous-time systems, like the Luenberger observer [6] or the Kalman filter [5], share one common property, namely, the system state is estimated in an asymptotic fashion. However, in some applications, e.g. in fault detection, it is desirable to estimate the system state in finite time.

Observers with predetermined finite convergence time have been developed in [1–4, 7] and the references therein. By using information of the past, e.g. delayed observer states [2], delayed measurements [1, 3, 7] or delayed innovation processes [7], these observers estimate the exact state of a linear continuous-time system in predetermined finite time. However, from a practical point of view these observers are rather difficult to realize since they require a large (infinite) amount of memory due to the storage of trajectory pieces [1–3, 7] and/or the instantaneous solution of convolution integrals over a finite time horizon [1, 3, 7].

This paper presents an observer with predetermined finite convergence time which avoids the above mentioned implementation problems. The finite convergence time of the proposed observer is achieved by updating the observer state based on current observer data at a definite time instant. This abrupt update of the observer state, which can be easily realized if the observer is implemented e.g. on a microcontroller, causes impulsive behavior of the observer dynamics, i.e. one obtains an impulsive observer. To the authors’ best knowledge the proposed observer structure is new in respect of observers for linear continuous-time systems with predetermined finite convergence time.

The remainder of the paper is organized as follows: In Section II, the main result of the paper is presented, namely, an observer that estimates the exact state of a linear continuous-time system in a predetermined finite time. Simulations are presented in Section III and Section IV concludes the paper with a summary and an outlook for some relevant future work.

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II. MAIN RESULT

Consider the linear continuous-time system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \quad (1a) \\
y(t) &= Cx(t), \quad (1b)
\end{align*}
\]

where \(x \in \mathbb{R}^n\) is the system state, \(u \in \mathbb{R}^p\) the input, \(y \in \mathbb{R}^q\) the output, and \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{q \times n}\) are constant matrices. It is assumed that system (1) is observable. The proposed observer consists of two Luenberger observers [6] with state updates at time instants \(t_k\), where \(0 \leq t_0 < t_1 < t_2 < \ldots < t_k < \ldots\) with \(\lim_{k \to \infty} t_k = \infty\), and an additional equation to generate the actual state estimate \(\hat{x}\), i.e.

\[
\begin{align*}
\dot{z}(t) &= Fz(t) + Gu(t) + Hy(t), \quad t \neq t_k, \quad (2a) \\
z(t^+_k) &= K_z z(t^-_k), \quad t = t_k, \quad (2b) \\
z(t^+_0) &= Mx_0, \quad k = 1, 2, \ldots \quad (2c) \\
\hat{x}(t) &= Nz(t). \quad (2d)
\end{align*}
\]

In (2), \(z = [z_1 \ z_2]^T\) with \(z_i \in \mathbb{R}^n\) is the observer state, \(\hat{x}_0 \in \mathbb{R}^n\) an estimate of the initial condition of system (1), \(z(t^+_k) = \lim_{h \to 0} z(t_k + h), z(t^-_k) = \lim_{h \to 0} z(t_k - h)\), and, without loss of generality [12], it is assumed that \(z(t^-_k) = z(t_k)\). The matrices in (2a), (2c), and (2d) are given by

\[
F := \begin{bmatrix} F_1 & 0_n \\ 0_n & F_2 \end{bmatrix}, \quad G := \begin{bmatrix} B \\ B \end{bmatrix}, \quad (3)
\]

\[
H^T := [L_1^T \ L_2^T]^T, \quad M^T := [I_n \ I_n]^T, \quad N := [I_n \ 0_n],
\]

where \(I_n\) is the \(n \times n\) identity matrix, \(0_n\) the \(n \times n\) zero matrix, \(L_i \in \mathbb{R}^{n \times d}\) matrices to be designed, and \(F_i := A - L_i C\). Moreover, the matrices \(K_k \in \mathbb{R}^{2n \times 2n}\) in (2b) are

\[
K_1 := M \left( I_n - e^{F_2 \delta e^{-F_1 \delta}} \right)^{-1} \left[ -e^{F_2 \delta e^{-F_1 \delta}} I_n \right],
\]

\[
K_k := I_{2n}, \quad k = 2, 3, \ldots
\]

with \(\delta = t_1 - t_0\). Note that the time instants \(t_k\), i.e. the time instants of the observer state updates, are design parameters which can be arbitrarily specified to achieve the desired convergence behavior of observer (2). The design parameters of observer (2) are the matrices \(L_1, L_2\) and the time instant \(t_1\). The next theorem states how to choose \(L_1, L_2, t_1\) such that observer (2) converges in finite time \(\delta\).

Theorem 1: Let the matrices \(L_1, L_2\) and the time instant \(t_1\) be designed such that matrix \(F\) is a Hurwitz matrix and that matrix \(K_1\) exists. Then equations (2) define an observer that estimates the exact state of system (1) in predetermined finite time \(\delta = t_1 - t_0\), i.e. \(\hat{x}(t) = x(t) \forall t > t_1\).
Proof. For $t_0 \leq t \leq t_1$ the estimation error dynamics of the two Luenberger observers in (2a) are
\[
\begin{align*}
\dot{z}(t) - \dot{z}_i(t) &= Ax(t) + Bu(t) \\
&\quad - (A\dot{z}_i(t) + Bu(t) + L_i(y(t) - Cz_i(t))) \\
&= F_i(x(t) - z_i(t)).
\end{align*}
\]
Hence, in the time interval $t_0 \leq t \leq t_1$ the observer states (2a) can be expressed depending on the system state (1) as
\[
z_i(t) = x(t) - e^{F_i(t-t_0)}(x_0 - \hat{x}_0),
\]
where $x_0$ is the initial condition of system (1) and $z_i(0) = \hat{x}_0$ the initial conditions of the Luenberger observers (2a). Using $z(t_1^+) = z(t_1)$ and $\delta = t_1 - t_0$, it follows from (2b), (6) that the observer state $z$ jumps at time instant $t_1$ to
\[
\begin{align*}
z(t_1^+) &= K_1[Mx(t_1) - \left[\begin{array}{cc} F_1 & e_{F_2}\delta \\ 0 & e_{F_2}\delta \end{array}\right] M(x_0 - \hat{x}_0)] \\
&= Mx(t_1)
\end{align*}
\]
due to the fact that
\[
K_1 = M \left( I_n - e^{F_2\delta e^{-F_1\delta}} \right)^{-1} \times \left[\begin{array}{cc} e^{F_1\delta} & 0_n \\ 0_n & e^{F_2\delta} \end{array}\right] \left( I_n - e^{F_2\delta e^{-F_1\delta}} \right).
\]
Hence, $\dot{x}(t_1^+) = N\dot{z}(t_1^+) = x(t_1)$ and $\dot{z}(t) = x(t)$ for $t > t_1$ since the Luenberger observers (2a) are initialized with the exact system state $x(t_1)$ at time instant $t_1$ due to (2b) and (7). This proofs that observer (2) estimates the exact state of system (1) in finite time $\delta$.

Remark 1: In Theorem 1 it is assumed that matrix $K_1$ exists. Since the pair $(A,C)$ is observable, the existence of $K_1$ can be guaranteed by an appropriate choice of the matrices $L_1, L_2$. Based on [2], choose the matrices $L_1, L_2$ such that
\[
Re\lambda_i(F_2) < \sigma < Re\lambda_i(F_1), \quad i = 1, \ldots, n
\]
with some $\sigma < 0$. Then it follows from (8) that the determinant of $\left( I_n - e^{F_2\delta e^{-F_1\delta}} \right)$ is not identically zero since $e^{F_2\delta e^{-F_1\delta}} = e^{-e^{-F_1\delta}(1-e^{-F_2\delta})} \neq 0$ for $\delta \rightarrow \infty$ and therefore $\det(\left( I_n - e^{F_2\delta e^{-F_1\delta}} \right)) \rightarrow 1$ for $\delta \rightarrow \infty$. Furthermore, $\det(\left( I_n - e^{F_2\delta e^{-F_1\delta}} \right))$ has only isolated zeros since it is an analytic function of $\delta$ [2]. This implies that $\det(\left( I_n - e^{F_2\delta e^{-F_1\delta}} \right))$ is nonzero for small values of $\delta$ since $\det(\left( I_n - e^{F_2\delta e^{-F_1\delta}} \right)) = 0$ for $\delta = 0$. Finally, one can conclude that $\left( I_n - e^{F_2\delta e^{-F_1\delta}} \right)^{-1}$ and therewith the matrix $K_1$ exist at least for sufficiently small values of $\delta$, i.e. for sufficiently small differences of $t_1 - t_0$.

Remark 2: It follows from Remark 1 that the convergence time $\delta = t_1 - t_0$ of observer (2) can be chosen almost arbitrarily fast.

Remark 3: The basic idea of observer (2) is the following one: Consider the estimation errors $e_i = x_i - z_i$ of the two Luenberger observers (2a) at time instant $t_1$, i.e.
\[
e_1(t_1) = x(t_1) - z_1(t_1)
\]
\[
e_2(t_1) = x(t_1) - z_2(t_1).
\]
From (6) it follows that $e_i(t_1) = e^{F_i(t_1-t_0)}(x_0 - \hat{x}_0)$. Hence, equation (9) can be rewritten as
\[
e_1(t_1) = e^{F_1(t_1-t_0)}(x_0 - \hat{x}_0) = e^{F_1\delta}e_0 = x(t_1) - z_1(t_1)
\]
\[
e_2(t_1) = e^{F_2(t_1-t_0)}(x_0 - \hat{x}_0) = e^{F_2\delta}e_0 = x(t_1) - z_2(t_1)
\]
with $e_0 = x_0 - \hat{x}_0$ and $\delta = t_1 - t_0$. By solving equations (10) for $x(t_1)$ one obtains the exact system state of (1) at time instant $t_1$. Hence, the finite convergence time of observer (2) is obtained by computing the actual state from (10) and initializing the two Luenberger observers (2a) with $x(t_1)$ at time instant $t_1$.

Remark 4: From a computational point of view the proposed observer (2) is more attractive than the finite time convergent observers [1–3, 7] since the memory requirements and the on-line computations are reduced. The reason for this is that these observers use information of the past, which have to be stored, and/or the on-line computation of convolution integrals in order to achieve finite time convergence whereas the finite convergence time of observer (2) is only achieved via observer state updates.

III. ILLUSTRATIVE EXAMPLE

Consider the following system
\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t),
\end{align*}
\]
\[
y(t) = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x(t)
\]
with $x \in \mathbb{R}^3$, initial condition $x_0 = [3, -5, -7]^T$, initial time $t_0 = 0$, and $u(t) = 0 \forall t \geq 0$. The parameters of observer (2), which are designed to satisfy the conditions of Theorem 1, are given by $L_1 = [2, -3, -1]^T$, $L_2 = [-107, 48, 23]^T$, and e.g. by $t_1 = 1.5$ with corresponding matrix $K_1 = \begin{bmatrix} 11.9 & 198.5 & 151.9 \\ -32.4 & -54.2 & -41.5 \\ -57.1 & -96.4 & -73.0 \end{bmatrix}$.

Fig. 1 shows the trajectories of system (11) and the trajectories of proposed observer (2) with initial condition $\hat{x}_0 = [0, 0, 0]^T$, $t_1 = 0.5$ (left), $t_1 = 1$ (center), and $t_1 = 1.5$ (right). Fig. 1 illustrates that the proposed observer converges exactly at the different time instants $t_1$ and that the convergence time $\delta = t_1 - t_0$ can be chosen almost arbitrarily, as discussed in Remark 1 and Remark 2.
IV. CONCLUSIONS

In this paper an impulsive observer with predetermined finite convergence time for linear continuous-time systems has been presented. It has been shown that the observer converges in finite time and that the convergence time can be chosen almost arbitrarily. Furthermore, some system theoretical properties of the proposed observer have been discussed and its finite time convergence behavior has been illustrated via an example. Similar to [8–11], future work intends to use the proposed observer structure to design observers with finite convergence time for linear time-varying systems, nonlinear systems, and linear systems with unknown inputs. Furthermore, future work should also study the performance of the proposed observer in presence of measurement noise and model uncertainty.

REFERENCES


