Unambiguous fault identification and accommodation for incipient and abrupt faults

G. Parlangeli, D. Pacella, M. L. Corradini

Abstract—This paper considers a failure model including both incipient and abrupt faults, and addresses the problem of identifying and accommodating such faults. An identification strategy is proposed for the considered fault class, and a test is presented able to discriminate between the two types of faults. In order to prove the effectiveness of the estimation procedure a fault tolerant control strategy based on those theoretical results has been designed and implemented by simulation on a vertical take-off and landing (VTOL) aircraft.

Index Terms—Actuator Failure, Unknown Input and State Estimation, Fault Tolerant Control.

I. INTRODUCTION

Controlling systems subject to faults is a challenging issue which is attracting more and more attention by control systems researchers. Indeed, failures of system components are unavoidable as a consequence of repetitive strain during their working activity. Automatic diagnosis and compensation for faults is therefore an important feature for a controlled system, since a faulty behavior of a system can jeopardize personnel and/or users involved, as for aeronautical systems (see, by example, [1], [13], [4] and references therein) or chemical plants ([17], [12] and more recently [14]).

An important issue for a fault diagnosis and accommodation algorithm is the early detection of faults, i.e. before their abnormal influence on the system can cause damage propagation, components breakdown and unpredictable (potentially catastrophic) system evolution. In the last decade, a large number of researchers focused on the specific problem of incipient fault diagnosis and, more recently, on their accommodation [15], [9], [21], [3]. In this framework, considered types of failures are either a sudden deviation from the nominal behavior of a component (the so called 'abrupt fault'), usually modelled by a step function, or a slowly varying degradation of the device characteristic (i.e. a monotone deterioration of the faulty component efficiency), known as 'incipient fault' [6], [7], [15].

According to the idea that a fault possibly causes both a sudden deviation from the nominal behavior of a component and a slowly varying efficiency degradation, as a consequence of the damage propagation in the device, this paper addresses the specific problem of identifying an unknown input belonging to a class of functions which model failures containing both abrupt and incipient terms; the identification procedure lasts a fixed pre-computed interval in order to avoid instability and it can be easily implemented and integrated into a fault tolerant control architecture.

In particular, in the last section of this paper the identification procedure is embedded into a fault tolerant control law. The control policy here proposed is based on sliding mode techniques, and has been proved to guarantee complete fault tolerance; the well assessed robustness features of sliding mode control have been exploited to ensure an effective failure accommodation. Theoretical results have been validated by a simulation study on a linearized model of a vertical take-off and landing (VTOL) aircraft [19], showing the effectiveness of the proposed algorithm. A few words about notation and the mathematical framework. It is assumed that the state space $X$ is a finite $n$-dimensional euclidean linear space ($\mathbb{R}^n$ denoting the zero vector) endowed with the usual inner product $\langle x, y \rangle = x^T y$ and induced norm: $\|x\| = (x^T x)^{\frac{1}{2}}$, $U$ denotes the set of the admissible input values at a time $t$, with the same properties of $X$ above; $Y$ is the set of the output function vectors (which, being solutions of differential equations in the Filippov sense, is the set of vectors of $p$ absolutely continuous functions) endowed with the norm $<y_a(\cdot),y_b(\cdot)>_{c_0}$ $= \int_0^\Delta y_a^T(\sigma)y_b(\sigma)d\sigma$. The adjoint operator of a linear operator over linear spaces $V,W$, $\omega : V \to W$, i.e. the linear operator $\omega^* : W \to V$ such that $<v,\omega^*(w)>_V = <\omega(v),w>_W \forall v \in V, \forall w \in W$ is denoted $\omega^*$. Vectors are usually written in bold characters; given a $n$-dimensional vector $z$ and two indexes $i,j \in \mathbb{N}$ such that $1 \leq i \leq j \leq n$ the notation $z_{[i:j]}$ represents the subvector $(j-i+1)$-dimensional composed of vector $z$ components indexed between $i$ and $j$; when $i=j$ the brackets are omitted, meaning the $i$-th component of a vector.

II. SYSTEM MODEL AND FAULT MODEL

Consider a dynamical system described by the equations:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector at time $t$, $u(t) = [u_1(t) \ u_2(t) \ \ldots \ u_m(t)]^T \in \mathbb{R}^m$ is the system input vector whose components $u_i(t) \in \mathbb{R}$, $i = 1, \ldots, m$ (m input) may fail during system operation, $B = [b_1 \ b_2 \ \ldots \ b_m] \in \mathbb{R}^{n \times m}$, with $b_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, is the input distribution matrix, $y(t) \in \mathbb{R}^p$ is the system output and $C \in \mathbb{R}^{p \times n}$ is the output distribution matrix.
It is assumed that a supervisor exists in charge of switching the input from the plant, if any, according to the results of the FDI device. Hence at any time, only one actuator is assumed active. For example, it is assumed that at the initial time \( t = 0 \) the first actuator is active (i.e. \( u_i = 0, \ i = 2, \ldots, n \)) and then actuators are checked sequentially. This actuator redundancy appears in system dynamics (1), and is exploited to accommodate faults ([16], [2] and references therein, [11], [10]).

Each input \( u_i(t) \) is not directly available for control since is the output of an actuator potentially affected by faults; failures reduce actuator effectiveness and faulty actuators affect the system entering as unknown disturbances. As a consequence, the \( i \)-th system input \( u_i(t) \) coincides with the controller output (denoted by \( v(t) \) hereafter) only before a fault occurs on the \( i \)-th actuator, the relationship between \( u_i(t) \) and \( v(t) \) being described by:

\[
u_i(t) = v(t) + \alpha_i(t)(u_{f_i} - v(t))\delta_{i-1}(t - t_f)
\]  

(2)

In equation (2) \( t_f \) is the time instant of the failure occurrence in the \( i \)-th actuator, \( u_{f_i} \in \mathbb{R} \) denotes the disturbing unknown input of a faulty actuator, while \( \alpha_i(t) \) is a positive real function varying within \((0, 1)\), and is related to the loss of effectiveness of the \( i \)-th actuator. Indeed, it describes how the occurrence of a fault modifies actuator performances, degrading them. If \( \alpha_i(t) \) is close to zero, the \( i \)-th actuator works close to nominal conditions, while when \( \alpha_i(t) \) is close to one, the \( i \)-th actuator performances are drastically reduced. In this latter case, the fault’s influence on the actuator output \( \alpha_i(t)u_{f_i} \) increases while the coefficient related to the control input \( v(t) \) vanishes. When \( \alpha_i(t) = 1 \), \( i \)-th actuator does not react to the control input \( v(t) \) at all. Often the loss of effectiveness is not constant but fault intensity gets worse reaching a steady state value \( \overline{\alpha}_i \) which can be significantly larger than its initial value. Therefore a mathematical model that describes the loss of effectiveness of the \( i \)-th actuator can be the following:

\[
\alpha_i(t) = \begin{cases} 
0 & \text{if } t < t_f, \\
\overline{\alpha}_i + (\beta_i - \overline{\alpha}_i)e^{-\frac{t}{\overline{\alpha}_i t_f}} & \text{if } t \geq t_f,
\end{cases}
\]

(3)

where \( \beta_i = \alpha_i(t_f) \) is the initial value of loss of effectiveness of the \( i \)-th actuator which models an abrupt change of the actuator characteristic. This model incorporates different situations including faults with constant loss of effectiveness (\( \overline{\alpha}_i = \beta_i \neq 0 \)) or purely exponential faults (\( \beta_i = 0 \)). Most of the times, in fact, a fault begins producing an abrupt variation in time, due to the malfunctioning of a component but, as time goes on and the failed component continues working, the fault effects grow worse. An easy way of modelling this behavior is by an (unknown) function exponentially converging to a steady state value of the loss of effectiveness, as described in (3), where \( 0 \leq \beta_i \leq \overline{\alpha}_i \leq 1. \) The term \( \theta_i > 0 \) (fault rate) denotes the rate of deterioration of the faulty component from its initial value to steady state.

The following hypotheses are assumed to hold.

Assumption 1: System (1) is reachable and observable as long as at least one actuator is failure free.

Assumption 2: An initial estimate of the state vector \( x_0 \) is available such that the estimation error \( e_{x_0} \) is bounded by a known constant \( |e_{x_0}| < \rho_{x_0} \).

Consider the system (1) satisfying Assumptions 1, 2, and subject to eventual faults modelled by (2) and (3). The problem here addressed is to identify the parameters of the faulty actuator in case a failure occurs, i.e. the determination of \( u_{f_i}, \alpha_i, \beta_i \) and \( \theta_i \) from the system output \( y(t) \) and controller output (which coincides with the actuator input) \( v(t) \).

III. IDENTIFICATION OF FAULT PARAMETERS

The following section is split into three parts. First, an identification rule is proposed for constant faults, then the same procedure is extended to include an exponentially varying fault model. In the last subsection, a test is presented able to discriminate between the two classes of models.

A. Identification of constant fault parameters

In this section a summary of the results available in [5], where constant faults are dealt with, are reported; the proofs are omitted for brevity, the definitions and results are reported for the sake of completeness of the next subsections.

For constant faults (\( \beta_i = \overline{\alpha}_i \)) equation (3) takes the simpler form \( \alpha_i(t) = \overline{\alpha}_i \). For identification purposes, output signal is recorded for time intervals of width of \( \Delta \); let \( \Psi \equiv [t_{d_1}, t_{d_1} + \Delta] \) where \( t_{d_1} \) is the detection instant of a fault of the \( i \)-th actuator. During such interval the controller output is set to zero, so the system input is unknown but constant, equal to \( u(t) = \overline{\alpha}_i u_{f_i} \); denote with \( \gamma_i(t) \) the effect on the output at time \( t \) of the already occurred and identified faults; let \( y_1(t') = y(t' + t_{d_1}) - \gamma_i(t' + t_{d_1}) \). Separating known signal and unknown parameters:

\[
y_1(t') = \begin{bmatrix}
C e^{A t'} & [T_{t_{d_1}} C e^{A(t'-\sigma)}] & [x(t_{d_1})] \overline{\alpha}_i u_{f_i}
\end{bmatrix}
\]

(4)

Signal \( y_1(t'), t' \in [0, \Delta] \) can be considered as the image of a linear operator defined on \( \mathbb{R}^{n+1} \) to \( C_{\mathbb{R}}^p \setminus \omega_c \) say \( \omega_c \). Consider the adjoint operator of \( \omega_c \); by direct calculation one gets:

\[
\omega^*_c(y_c) = \begin{bmatrix}
\int_0^\sigma C e^{A(t-\sigma)} C^T \Theta_1(\sigma) y_c(\sigma) d\sigma
\end{bmatrix}
\]

(5)

where \( \Theta_1(\sigma) = \int_0^\sigma C e^{A(t-\sigma)} b_1(\sigma) d\sigma \) is the step response of the system at time \( \sigma \).

The linear operator obtained by the composition \( \omega^*_c \left( \omega_c \left( \begin{bmatrix} \Theta_1 \in \mathbb{R}^p \nabla_c \in \mathbb{R}^n \end{bmatrix} \right) \right) \), is, by definition of \( \omega_c \), from \( \mathbb{R}^{n+1} \) to \( \mathbb{R}^{s+1} \) and it can equivalently be expressed by a (time-varying) square matrix \( W_c(\Delta) \); by direct inspection:

\[
W_c(\Delta) = \begin{bmatrix}
w_{11} & w_{12} & w_{12} \\
w_{12} & w_{12} & w_{22}
\end{bmatrix}
\]

where

\[
w_{11} = \int_0^\Delta e^{A(t-\sigma)} C e^{A(t-\sigma)} d\sigma; \quad w_{12} = \int_0^\Delta e^{A(t-\sigma)} C e^{A(t-\sigma)} \Theta_1(\sigma) d\sigma;
\]

\[
w_{22} = \int_0^\Delta \Theta_1(\sigma)\Theta_1(\sigma) d\sigma.
\]

Theorem 3.1: Consider the plant (1) under the hypothesis (1), det\( W_c(t) \neq 0 \) \( \forall t > 0 \) for \( i = 1, \ldots, m \).
Proposition 3.1: Unknown parameters vector can be computed by
\[ \tilde{\zeta}_c \triangleq \begin{bmatrix} x(t_{d1}) \\ \pi_t u_{f1} \end{bmatrix} = W_1(\Delta)^{-1} \omega^*_c(y_1(t')) \] (6)

The proof is omitted for the sake of brevity, the interested reader is referred to [5].

B. Identification of exponential faults

The identification procedure for unknown exponential inputs is more involved. First of all the influence of faults previously identified from the output signal \( y(t), \gamma_i(t) \), is preliminarily subtracted. Consider two time intervals denoted:

- \( \Psi_1 \) if \( t \in [t_{d1}, t_{d2} + \Delta] \);
- \( \Psi_2 \) if \( t \in [t_{d2} + \Delta, t_{d3} + 2\Delta] \).

In order to make some preliminary considerations, set \( v(t) = 0 \) during these intervals; system input (equations (2) and (3)) becomes:
\[ u_i(t) = \pi_i u_{f1} + (\beta_i - \pi_i) u_{f1} e^{-\frac{t-t_{d1}}{\alpha}} \]

To simplify the notation set:

- \( t' = t - t_{d1} - \Delta \Rightarrow t = t' + t_{d1} + \Delta \) con \( t' \in [0, \Delta] \);
- \( y_1(t') = y(t' + t_{d1}) - \gamma_i(t' + t_{d1}), \) system output in \( \Psi_1 \);
- \( y_2(t') = y(t' + t_{d1} + \Delta) - \gamma_i(t' + t_{d1} + \Delta), \) system output in \( \Psi_2 \);

Computing system output in \( \Psi_1 \) and \( \Psi_2 \), these relations can be joined together giving the following equation:
\[ y_2(t') = \left[ Ce^{At'} : \int_0^{t'} Ce^{A(t' - \sigma)} b_i d\sigma \right] \begin{bmatrix} y_1(t') \\ 0 \end{bmatrix} \] (7)

Following the same lines as before one can focus his attention on determining the vector:
\[ \tilde{\zeta}_c = \begin{bmatrix} x(t_{d1} + \Delta) - x(t_{d1}) e^{-\frac{\Delta}{\alpha}} \\ \pi_t u_{f1} (1 - e^{-\frac{\Delta}{\alpha}}) \\ e^{-\frac{\Delta}{\alpha}} \end{bmatrix} \] (8)

from which it is possible to compute fault parameters and an estimation of \( x(t_{d1}) \).

The adjoint operator of the linear application which links signal \( y_2(.) \) to \( \zeta_c \) in (7) takes the form:
\[ \omega^*_c(y_2(.)) = \begin{bmatrix} \int_0^{\Delta} e^{A^T \sigma} C^T y_2(\sigma) d\sigma \\ \int_0^{\Delta} \Theta_1(\sigma) y_2(\sigma) d\sigma \\ \int_0^{\Delta} y_1(\sigma) y_2(\sigma) d\sigma \end{bmatrix} \] (9)

Applying to both sides of (7) the adjoint operator one obtains the relation \( W_{a_1}(\Delta) \zeta_c = \omega^*_c(y_2(t')) \) where \( W_{a_1} \) is a square \((n+2)\)-order matrix whose structure is:
\[ W_{a_1}(\Delta) = \begin{bmatrix} W_1(\Delta) & w_{14} & w_{15} \\ w_{14} & w_{24} & w_{25} \\ w_{15} & w_{25} & w_{35} \end{bmatrix} \] , with:
\[ w_{13} = \int_0^{\Delta} e^{A^T \sigma} C^T y_1(\sigma) d\sigma; \]
\[ w_{33} = \int_0^{\Delta} y_1(\sigma) y_1(\sigma) d\sigma. \]

If \( W_{a_1}(\Delta) \) were invertible the unknown vector of fault parameters could be computed by:
\[ \tilde{\zeta}_c = W_{a_1}(\Delta)^{-1} \omega^*_c(y_2(t')). \] (10)

An interesting result is described in the next subsection and it states that the invertibility of matrix \( W_{a_1}(\Delta) \) is strictly related to the time evolution of the fault and it allows to distinguish between constant and exponential faults as described in the following section.

C. Relation between invertibility of \( W_{a_1}(\Delta) \) and fault time evolution

It is now useful to focus the attention on the invertibility of the matrix \( W_{a_1}(\Delta) \). It will be shown in the following that this issue allows to discriminate between constant and exponential fault by a test on the plant output during the first interval.

It is worth noting that special care has to be used when the fault rate satisfies \(-\theta^{-1} = s^*, s^* \) being a zero of system (1). This special case will be addressed at the end of the section.

Recall that in the time domain [8] \( s = s^* \) is a system zero if and only if there exist vectors \( \tilde{x} \in \mathbb{C}^n \) and \( \tilde{u} \in \mathbb{C}^m \), \( \tilde{u} \neq 0 \), such that
\[ Ce^{At}\tilde{x} + \int_0^t Ce^{A(t-\sigma)} B\tilde{u} e^{s^*\sigma} d\sigma = 0 \] (11)

Moreover, it is easy to show that, if \( s^* \in \mathbb{R} \), a necessary and sufficient condition for (11) to hold is the existence of real vectors \( \tilde{x} \in \mathbb{R}^n \) and \( \tilde{u} \in \mathbb{R}^m \) satisfying (11). In the single input case this relation implies that, when \( u(t) \) takes the form (and only in this case) of \( u(t) = \tilde{u} e^{s^*t} \), \( s^* \) a real system zero, then the forced output \( y_f(t) = \int_0^t Ce^{A(t-\sigma)} B\tilde{u} e^{s^*\sigma} d\sigma \) always coincide with the free evolution of the system for a suitable initial condition \( \tilde{x} \), \( \exists \tilde{x} \in \mathbb{R}^n : y_1(t) = Ce^{At}\tilde{x} = y_f(t) \ \forall t \). In fact, being \( \tilde{u} \) a nonzero number satisfying (11), there always exists a (unique) \( \zeta \) such that \( \tilde{u} = \zeta \tilde{u} \). Multiply both sides of (11) by \( \zeta \) and choose \( \tilde{x} = -\zeta \tilde{x} \) to prove the statement.

The next result shows that the singularity of matrix \( W_{a_1}(t) \) allows to discriminate between constant and exponential fault behavior, provided that \(-\theta^{-1} \neq s^* \).

Theorem 3.2: Assume that \(-\theta^{-1} \neq s^* \) is not an invariant zero of the subsystem \((A, b_i, C)\). Then, under the Assumptions (1) and (4), a fault is constant \( \Leftrightarrow W_{a_1}(\Delta) \) is singular.

Proof: Consider the system dynamics with an unknown constant input. Relation (5) can be written \[ \begin{bmatrix} w_{13} \\ w_{23} \end{bmatrix} = \omega^*_c(y_1(.)) \]. Since the output of the system fed by a constant input is given by (4), the above equations can be coupled providing \( w_{33} = w_{13}^T x(t_{d1}) + \pi_t u_{f1} \). As the vector of fault parameters satisfies \[ \begin{bmatrix} x(t_{d1}) \\ \pi_t u_{f1} \end{bmatrix} = W_{a_1}(\Delta)^{-1} \begin{bmatrix} w_{13} \\ w_{23} \end{bmatrix} \] , a constant fault is such that: \( w_{33} = [ w_{13}^T \ w_{23} ] W_{a_1}(\Delta)^{-1} [ w_{13} \ w_{23} ] \).

On the other hand, partitioning into blocks the matrix \( W_{ia}(\Delta) \), one gets
\[ \det[W_{a_1}(\Delta)] = \det \begin{bmatrix} W_{11}(\Delta) & w_{13} & w_{15} \\ w_{14} & w_{24} & w_{25} \\ w_{15} & w_{25} & w_{35} \end{bmatrix} = \]
The previous result showed that testing \( w_{33} = [ w_{13}^T \quad w_{23} ] W_i(\Delta)^{-1} \begin{bmatrix} w_{13} & w_{23} \end{bmatrix} \) at the end of the first interval allow to check whether the fault has a constant time evolution or it increases in time, but only if \( -\theta_i^{-1} \neq s^* \).

Indeed, it is easy to show that, when \( -\theta_i^{-1} = s^* \), \( x(t) \) is not injective and this implies that \( W_i(\Delta) \) is singular for each \( \Delta \in \mathbb{R}_+ \). This statement can be proved showing that there exist a nonzero vector of initial conditions \( \bar{x}_0 \) and fault parameters \((\alpha_0, \beta_0, \bar{u}_f)\) giving an identically zero output.

Choose \( \alpha_0 = 0, \beta_0 f_j \) arbitrarily (nonzero) and \( x(t_d) = x_0 = \frac{\beta u_f e^{-\Delta}}{\hat{u}} \bar{x} - \beta u_f \int_0^\Delta e^{-\sigma} b e^{-\hat{\theta}_i^*} d\sigma \), \( \bar{x} \) and \( \hat{u} \) as in (11). In this case \( y(t) = 0 \) when \( t \in \mathcal{P} \) and nonzero input \( u(t) = \hat{u} e^{-\hat{\theta}_i^*} \) is applied; based on this consideration it is easy to see, by linearity, that system dynamics supplied by a constant fault \((\bar{x}, \bar{x}, \bar{u}_f)\) or by a fault with fault rate \(-\theta_i^{-1} = s^*\) and \((\bar{x} + x_0, \bar{x}, \bar{u}_f)\) show the same output evolution, so the two faults are indistinguishable from the output evolution.

One can conclude that, if \( W_i(\Delta) \) is singular, then it is possible that either a constant fault or an exponential fault characterized by \(-\theta_i^{-1} = s^*\) is affecting the controlled system. In the latter case, using (6), one is able to identify correctly \( \alpha_i u_f \), but one would wrongly determine the ‘fictitious’ initial value \( x_{i,a} = x_0 - (\beta_i - \alpha_i) u_f \bar{x} \) instead of \( x_d \). This happens because, in view of (11), \( y(t) = Ce^{A(t-t_d)} x_d + \alpha_i u_f \int_0^{t-t_d} Ce^{A(t-t_d-\sigma)} \beta_d \sigma + (\beta_i - \alpha_i) u_f \int_0^{t-t_d} Ce^{A(t-t_d-\sigma)} \beta_d e^{-\hat{\theta}_i^*} d\sigma = Ce^{A(t-t_d)} (x_d - \bar{x}) + \alpha_i u_f \int_0^{t-t_d} Ce^{A(t-t_d-\sigma)} \beta_d d\sigma \).

\[ D. \text{Avoiding ambiguity} \]

The discussed ambiguity can be avoided performing an identification experiment (of arbitrary duration) using a suitable control input. Use equation (6) at the end of the first time interval (computing, say, \( x_{i,a} \) and \( \alpha_i u_f \)) and set \( v(t) = e^{-(t-t_d-\Delta)} \) (that is, choose to set the control input equal to \( e^{-\hat{t}} \) during the time interval \([t_d + \Delta, t_d + 2\Delta])\).

System output for \( t \in [t_d + \Delta, t_d + 2\Delta] \) is equal to

\[ y(t) = Ce^{A(t-t_d)} x_d + \alpha_i u_f \int_0^{t-t_d} Ce^{A(t-t_d-\sigma)} \beta_d \sigma + (\beta_i - \alpha_i) u_f \int_0^{t-t_d} Ce^{A(t-t_d-\sigma)} \beta_d e^{-\hat{\theta}_i^*} d\sigma + \int_0^{t-t_d} Ce^{A(t-t_d-\sigma)} \beta_d [1 - \alpha_i(\sigma)] e^{-(\sigma-\Delta)} d\sigma. \]

Performing plant output reconstruction as if the fault were constant one has:

\[ y_{rec}(t) = Ce^{A(t-t_d)} x_{i,a} + \alpha_i u_f \int_0^{t-t_d} Ce^{A(t-t_d-\sigma)} \beta_d d\sigma + \int_0^{t-t_d} Ce^{A(t-t_d-\sigma)} \beta_d [1 - \alpha_i(\sigma)] e^{-(\sigma-\Delta)} d\sigma, \]

where the last term is unpredictable because \( \alpha_i \) (only) is unknown. Setting this last term to zero (which corresponds to assigning an a-priori value to \( \alpha_i = 1 \)) and computing \( \tilde{y}(t) = y(t) - y_{rec}(t) \) one has:

\[ \tilde{y}(t) = (1 - \alpha_i) \int_0^{t-t_d} Ce^{A(t-t_d-\sigma)} \beta_d e^{-(\sigma-\Delta)} d\sigma - (\beta_i - \alpha_i) \int_0^{t-t_d} Ce^{A(t-t_d-\sigma)} \beta_d e^{-(\sigma+\hat{\theta}_i^*)} d\sigma. \]

Assume that \( p \geq 3 \); then, computing the above signal at \( t = \Delta \), one has

\[ \{ \tilde{y}_1(\Delta) = (1 - \alpha_i)[\Theta_i(\Delta) - \beta_i - \alpha_i][\Theta_i(\Delta)]_1, \tilde{y}_2(\Delta) = (1 - \alpha_i)[\Theta_i(\Delta) - \beta_i - \alpha_i][\Theta_i(\Delta)]_2, \tilde{y}_3(\Delta) = (1 - \alpha_i)[\Theta_i(\Delta) - \beta_i - \alpha_i][\Theta_i(\Delta)]_3 \} \]

where \( \Theta_i(t) \) is computed solving \( n - 1 \) different linear systems of the first two equations in the two parameters \((\alpha_i \text{ and } \beta_i)\) for \( \ell = 1, 2, \ldots, n - 1 \) with \( \theta_i^* \) zero of system, and then using the third relation as a ‘parity relation’ in order to find an eventual hidden exponential. If the fault is constant each solution shows the equality \( \beta_i = \alpha_i \). If the system has a scalar output, the same approach can be adopted but, in order to have three independent linear relations in \( \alpha_i \text{ and } \beta_i \), one should measure the output at three different time instants, for example \( t_1 = t_d + \Delta + \frac{\Delta}{3}, t_2 = t_d + \Delta + \frac{2\Delta}{3} \text{ and } t_3 = t_d + 2\Delta \).

Concluding, the previous considerations show that the following unique strategy can be applied for fault parameter identification. Assume that a fault has been detected at time \( t_d \):

- Set \( v(t) = 0 \) and record system output \( y(t) \) within the interval \([t_d, t_d + \Delta] \).
- If \( \det[W_i(\Delta)] \neq 0 \) then set \( v(t) = 0 \), record \( y(t) \) within the interval \([t_d + \Delta, t_d + 2\Delta] \) and use relation (10) to compute \( \tilde{\zeta}_c \).
- If \( \det[W_i(\Delta)] = 0 \) use relation (6) to compute \( \tilde{\zeta}_c \) and set \( v(t) = e^{-\hat{t}} \), then record \( y(t) \) within the interval \([t_d + \Delta, t_d + 2\Delta] \), use relation (12) to detect an eventual \(-\theta_i^{-1} = s^* \), \( s^* \) a real system zero.

In any case, the estimation procedure lasts \( 2\Delta \).

\[ E. \text{Computation of each parameter} \]

After the occurrence of a constant fault, the unknown fault parameters are given by the vector \( \zeta_c \) in (6). Conversely, in the case of unknown exponential faults, a further step is required to derive each parameter from the knowledge of \( \zeta_c \), their relation being expressed by (8). In fact, after the detection of an exponential fault and the computation of the vector \( \zeta_c \) by (10), the final issue is the derivation of each single parameter. This is not a trivial task, but it can be achieved making resort to an augmented state space. Consider a new state variable \( x_{n+1}(t) = \left( \begin{array}{c} x_{n+1}(t) \end{array} \right) \) whose dynamics is governed by the equation \( \dot{x}_{n+1}(t) = -\frac{1}{\beta_i} x_{n+1}(t) \). The estimation of the initial state of the augmented state \( \chi(t) = \begin{bmatrix} x(t) \\ x_{n+1}(t) \end{bmatrix} \).
leads to a formula for determining each fault parameter. The observability of the augmented system is thus related to the exponential rate.

**Proposition 3.2:** Under assumption (1), the output map $(\hat{A}, \hat{C})$ of the extended system is completely observable $\iff s^* = -\theta - 1$ is not a system zero.

**Proof:** Directly from applying the observability PBH test to the extended system.

In view of the observability of the augmented system, one has
\[
\chi(t_{di}) = \int_0^{\Delta} \exp \left[ \int_0^{\tau_d(t)} \exp \left( -\hat{A} \tau_d(t) - \hat{C} \hat{d} \xi \right) \right] d\sigma.
\]

where $\hat{A} \overset{\text{def}}{=} \begin{bmatrix} A & b_1 \\ 0 & 1 \end{bmatrix}$, $\hat{B} \overset{\text{def}}{=} \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$, $\hat{C} \overset{\text{def}}{=} \begin{bmatrix} C & 0 \end{bmatrix}$

are the system matrices of the augmented state space of the system, and $\gamma_\Delta$ is the observability gramian of the extended system [18]. Using this result, fault parameters and state vector at time $t_{di}$ can be computed as follows:

\[
\theta_i = -\frac{\Delta}{\ln(\zeta_{n+2})}, \quad \pi_i u_{fi} = \frac{\zeta_{n+1}}{1 - \zeta_{n+2}}, \quad \beta_i, u_{fi} = \chi_{n+1}(t_{di}) + \pi_i u_{fi}, \quad x(t_{di}) = \chi_{1:n}(t_{di})
\]

and finally an information necessary to re-initialize the observer as described in the next section.

**IV. SIMULATION RESULTS**

In order to validate the previous estimation procedure it has been integrated within a fault tolerant control based on variable structure control described in the following, simulation have been made on a linearized model of a vertical take off and landing (VTOL) aircraft. When no actuator is faulty:

\[
v(t) = -(S_1 C b_1)^{-1} S_1 C A \hat{x}(t) - \bar{\varepsilon}(t) \text{sgn}(S_1 y(t)) \quad (14)
\]

\[
\dot{\hat{x}}(t) = A \hat{x}(t) + B u(t) - L(y(t) - \hat{y}(t)), \quad \hat{x}(0) = \xi_0 \quad (15)
\]

\[
\bar{\varepsilon}(t) = \varepsilon + \sum_{i=1}^n \left( S_1 C A \right) \rho x(t) \quad (16)
\]

where $\varepsilon$ is a positive arbitrary design constant related to the maximum reaching time of the sliding surface $T_R = \frac{|S_1 C x(0)|}{|S_1 C x(0)|}$. The fault detection rule which enables the identification procedure is given by a continuous monitoring of the sliding surface after the reaching time has elapsed.

After the fault has been identified, its accommodation requires the update of the state estimator and controller structure. As soon as $i$-th fault has been identified, the fault vector $\Xi_f$ storing fault parameters must be updated accordingly $(\Xi_f)_i(t) \overset{\text{def}}{=} \alpha_i(t) u_{fi}$. Finally, the next available control input (say the $k$'-th) is activated, exploiting the inherent redundancy of the system; to manage the presence of the disturbing input caused by the faults such actuator is fed by the control law (14),(15) adding the term $-(S_k b_n)^{-1} S_k \Xi_f(t)$ in the right hand side, thus accounting for fault compensation by the active $k$-th actuator.

\[
v(t) = -(S_1 C b_1)^{-1} S_1 C A \hat{x}(t) - \bar{\varepsilon}(t) \text{sgn}(S_1 y(t)) - (S_k b_n)^{-1} S_k B \Xi_f(t)
\]

and the initial state vector must be reconstructed accordingly to the value of the estimated state and input values; in the case of constant fault the initial condition of the state should be set: $\hat{x}_0 = e^{\Delta} z \hat{C}_{[1:n]} + \int_0^{\Delta} e^{\Delta} B \Xi_f(s) ds$ and, analogously, in the case of exponential faults:

\[
\hat{x}_0 = e^{\Delta} z \hat{C}_{[1:n]} + (1 - e^{\Delta}) B \Xi_f(t) \quad (16)
\]

The above control law ensures the achievement of a sliding mode motion on $S_k = \{ x \in \mathbb{R} : S_k x = 0 \}$ within the time $T_R = |S_k C x(0)|$ and, moreover, the sliding surface can be monitored for detecting further faults.

The linearized model of a VTOL implemented in the simulation is ([19], [20]):

\[
A = \begin{bmatrix}
-9.4774 & -0.7476 & 0.2632 & 5.0337 \\
52.1659 & 2.7452 & 5.532 & -24.4221 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} = \begin{bmatrix}
0.4422 & 0.1761 \\
3.5446 & -7.5922 \\
-5.52 & 4.49 \\
0 & 0
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

State components representing: $x_1$ horizontal speed of the vehicle; $x_2$ vertical speed of the vehicle; $x_3 = y_1$ yaw angle rate; $x_4 = y_2$ yaw angle.

State initial value is set equal to $x_0 = [1 1 1 1]^T$ and its initial estimate $\hat{x}_0 = [1.3 0.4 0.8 0.9]^T$, $\rho x_0$ being set to 1. Controller parameters have been chosen in order to ensure sliding motion within $t \leq 0.855s$.

For the sake of brevity only one simulation is reported; parameters describing the fault have been set equal to: $l_f = 5; \alpha = 0.6230; \beta = 0.213; \theta = 12; u_f = 16$ (so $\sigma u_f = 9,968, \beta u_f = 3,41$).

Fault detection and identification results are the following:

- detection time: $t_d = 5.015$ seconds;
- estimation of $\pi u_f = 9.9681$;
- estimation of $\theta$ equal to 12,004;
- estimation of $\beta u_f$ equal to 3.4148;

In Fig. 1 the phase plane is shown while Fig. 2 reports the control law and the three steps (control of fault-free system, fault identification, fault compensation) are clearly distinguishable. Fig. 3 shows the time evolution of the fault, the fault estimation and the system output.

**V. CONCLUSIONS**

In this paper, an unknown input and state estimator is proposed for input functions belonging to a given class in order to model both incipient and abrupt faults. In order to prove the effectiveness of the estimation procedure the proposed algorithm is put as block of a fault tolerant control strategy in a simulation on a VTOL, showing encouraging results.


