A simple hybrid/switching strategy for improving linear controllers

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Abstract—This paper presents a method for improving linear controllers once they have been designed using any linear technique. The controller enhancement is achieved by the introduction of a block which varies two parameters according to the error amplitude. It is also presented a stability condition for the designs, based on previous stability results for switching systems. Both the design method and the stability criterion are easy to use for a control engineer, even not being a specialist on switching control. A classical two-mass-spring benchmark control problem is also presented to illustrate the new technique.

I. INTRODUCTION

Although the literature in nonlinear control is immense, most of it is devoted to control nonlinear systems by using linear controllers. Usual tools like describing function, phase plane analysis, Popov criterion, etc, are applied to analyze system nonlinearities while a linear controller is designed. Its objective is often that the system behaves like a linear one, avoiding undesirable effects like limit cycles, jump resonances or subharmonic oscillations.

Conversely, this paper focuses on the advantages that can be obtained by using nonlinear compensators to control linear systems. The interest is addressed specifically to compensators designed by any linear method which are later modified with static nonlinear elements. Doing this, it is expected that the new nonlinear character of the system will leave it beyond performance limitations inherent to linear systems [1].

This article is directly inspired by Boris Lurie’s work [2], which deals with nonlinear elements (usually saturations and dead zones) for real-world control applications. Nonlinear static elements provide the system global stability and also improve its frequency domain characteristics, breaking the performance limitations of linear systems (Bode’s limitations). Particularly interesting are what Lurie names “multiwindow controllers”, that is, controllers that are divided in parallel blocks or windows, each one dominant in a frequency spectrum. Before each linear block it is placed a static nonlinearity which links its effect to a certain range of error amplitude. For this purpose, the nonlinearity is replaced by its describing function. Fig. 1 shows a simple two-window compensator.

Another use of the nonlinear elements is presented in [3], where performance specifications were unachievable with a linear controller. After designing a linear block, a lead compensator is extracted and its equation is modified with an odd nonlinear term multiplied by a coefficient. The controller describing function is computed by using Volterra series, allowing the designer to consider the system in the frequency domain. With a suitable value for the coefficient the system accomplishes the specifications. As it is stated in the conclusions of that paper, “the nonlinearities serve as a mechanism in which to break the Bode gain/phase relationship thereby improving closed-loop performance.”

The main drawback of this sort of techniques is that, although often effective, they are not properly an exact calculation, as a consequence of the assumptions that the describing function method requires. Moreover, stability is not assured for any error amplitude, that is, a system working correctly for some ranges of error amplitude may become unstable as a consequence of a big disturbance or a large reference change.

On the contrary, this paper presents a strategy where the nonlinear elements are not placed in the same path that the control signal, but in a sort of parallel path, which is actually a switching circuit. Then the whole system can be analyzed as a hybrid/switching one, and does not need a tool like the describing function, with the assumptions that it entails.

In that context some authors [4-5] have been advising about the performance benefits that hybrid/switching control may have in control systems. However, most of the literature about hybrid/switching control is still devoted to stability issues. The nonlinear scheme presented in this paper can be viewed as a linear switching controller in which a pole and a zero vary...
according to a nonlinear function that depends on the error amplitude. Stability of the design is guaranteed by a new result based on the stability radii concept.

The strength of this paper is that it provides a powerful but simple methodology to improve the performance beyond the limitations of linear systems. The technique is useful for a classical control engineer when his linear design has already been performed.

II. DESIGN TECHNIQUE

The main objective of the design technique is to speed up the system response when the error signal is large, though making it less accurate, and to invert the characteristics when the error signal is small. Those concepts are clearly explained from the classical control point of view in [2], where the behaviour of the control system is divided into two regimes (acquisition and tracking) depending on the error amplitude.

Let us consider a closed loop system whose plant and controller transfer functions are \( P(s) \) and \( G(s) \) respectively, and assume that the system is stable. Let \( L(s) \) then be the open loop transfer function, \( L(s) = P(s)G(s) \), and \( L'(s) \) the remaining function when a pole-zero block of the original controller is separated and put before in the loop. Fig. 2 shows the system.

Now the key point is to modify the zero-pole block with nonlinear elements, as it was done in [3]. However, in the present paper the modification is made on the block diagram of a zero-pole or phase lead-lag block. This block diagram is shown in Fig. 3. As can be seen, it is a command feedforward scheme which introduces a real pole on \( s = -p_1 \) and a real zero on \( s = -z_1 \).

The design of the nonlinear functions \( N_1 \) and \( N_2 \) is based on frequency domain and root locus considerations. Usually, the best results are obtained moving the zero to the right and the pole to the left when the error amplitude is large, and doing the inverse when it is small. Thus, for large error signals, the system is made faster and more stable, though more sensible to sensor noise. Conversely, when the error reduces, the system is slower and less stable, but the steady state error and the sensitivity to noise diminish.

The transition between the two regimes can be smooth or abrupt. It is a designer’s task to choose the nonlinear functions \( N_1 \) and \( N_2 \) which maximize the benefits of the variable controller. As it is proved in the next section, stability is assured for any switching signal which moves the parameters within a specific domain. The only condition is that the function must be piecewise continuous.

The designer needs some knowledge about the amplitude of reference changes and disturbances. Otherwise, the system performs like the one of the extreme systems during most of the response time, and little advantage is obtained from the method.

So the design method could be summarized in

1. controller design, using any linear technique,
2. selection of the pole \( p_1 \) and the zero \( z_1 \) that will be split up from the controller,
3. calculation of the variation range \((\Delta z_1, \Delta p_1)\) for the two parameters \((z_1, p_1)\),
4. design of the nonlinear functions \((N_1, N_2)\), using all the above considerations, and
5. implementation in a block diagram similar to Fig. 4.
III. STABILITY RESULTS

There are a lot of results in stability for switched linear systems, though no one is general. However, it has been proved [6] that a system
\[ \dot{x}(t) = A(t)x(t), \quad A(t) \in \mathcal{A} = [A_1, ..., A_m], \quad A_i \text{ Hurwitz}, \] (1)
with arbitrary switching within the set of matrices is exponentially stable if and only if there exists a common Lyapunov function (CLF) for all \( A_i \) in the set \( \mathcal{A} \). It has been also proved that the existence of a common quadratic Lyapunov function (CQLF) is a sufficient condition for exponential stability.

As a consequence, some researchers have found some results about the conditions under which the existence of a CQLF is assured. A number of them are based on the conditions under which the existence of a CLF is assured. A number of them are based on the stability radius concept, which is essentially the maximal perturbation in the parameters that does not destroy the exponential stability. In this context, linear switching systems take the structure
\[ \dot{x}(t) = (A + B\Delta(t)C)x(t), \] (2)
with \( B \in \mathbb{R}^{n \times d}, C \in \mathbb{R}^{d \times n} \), and where \( \Delta(t) \) is an unknown perturbation known as the perturbation matrix. The theorems relate spectral norms of the perturbation matrix with stability, so the first step is to obtain an expression for the system of Fig. 2 in terms of (2).

Let the transfer function \( L'(s) \) be
\[ L'(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + ... + b_0}{s^n + a_{n-1} s^{n-1} + ... + a_0}, \] (3)
so the closed loop transfer function \( T(s) = Y(s)/R(s) \) is
\[ T(s) = \frac{s + z_1}{s^n + a_{n-1} s^{n-1} + ... + a_0} + \frac{\Delta}{s + p_1 + \Delta p_1} + \frac{h_1}{s + z_1} + h_1(z_1 + \Delta z_1), \] (4)
and the characteristic equation
\[ (s + z_1)(s^n + a_{n-1} s^{n-1} + ... + a_0) + (s + p_1)(s^n + a_{n-1} s^{n-1} + ... + a_0) = 0, \] (5)
which can be written in the form
\[ s^{n+1} + c_n s^n + ... + c_1 s + c_0 = 0, \] (6)
where for each \( i = 0 ... (n+1) \)
\[ c_i = a_i - 1 + p_1 + h_1 + h_2 z_1. \] (7)

The controllable canonical form of the system has the state matrix
\[ A = \begin{bmatrix} 
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
-c_0 & -c_1 & -c_2 & \cdots & -c_{n-1} & -c_n 
\end{bmatrix}. \] (8)

If the phase lead/lag element is replaced by
\[ G(s) = \frac{s + z_1 + \Delta z_1}{s + p_1 + \Delta p_1}, \] (9)
then the characteristic equation is the same, but the coefficients are determined by
\[ c_i = a_i - 1 + p_1 (p_1 + \Delta p_1) + h_1 (z_1 + \Delta z_1), \] (10)
that is equal to
\[ c_i' = c_i + a_i \Delta p_1 + b_1 \Delta z_1. \] (11)

So the new state matrix can be divided as follows
\[ A' = A + \begin{bmatrix} 0 & & & & \Delta \beta_1 & \Delta \beta_1 & \cdots & \Delta \beta_1 \\
-\Delta \beta_1 & 0 & & & -\Delta \beta_1 & -\Delta \beta_1 & \cdots & -\Delta \beta_1 \\
0 & 0 & \cdots & \cdots & \Delta \beta_1 & \Delta \beta_1 & \cdots & \Delta \beta_1 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-\Delta \beta_1 & -\Delta \beta_1 & \cdots & -\Delta \beta_1 & 0 & & & \end{bmatrix}, \] (12)
where the second term can be written as a matrix product, that is,
\[ A' = A + \Delta \beta_1 \begin{bmatrix} 0 & \cdots & \Delta \beta_1 \\
0 & \cdots & \Delta \beta_1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \Delta \beta_1 \\
1 & \cdots & \Delta \beta_1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \Delta \beta_1 \\
0 & \cdots & \Delta \beta_1 \\
0 & \cdots & \Delta \beta_1 \\
\end{bmatrix}. \] (13)

Now the stability results compiled in [6] can be applied. Three stability radii are defined: \( r_k(A, B, C) := \inf \{ ||\Delta_0||_2 | \Delta_0 \in \mathbb{R}^{k \times q} \} \) is not exp. stable for \( \Delta(t) = \Delta_0 \), (14)
\[ r_k(A, B, C) := \inf \{ ||\Delta_0||_\infty | \Delta_0 \in \mathbb{R}^{k \times q} \} \] is not exp. stable for \( \Delta(t) = \Delta_0 \), (15)
\[ r_k(A, B, C) := \inf \{ ||\Delta_0||_\infty | \Delta_0 \in \mathbb{C}^{k \times q} \} \] is not exp. stable for \( \Delta(t) = \Delta_0 \), (16)
satisfying the relation
\[ r_k(A, B, C) \leq r_k(A, B, C) \leq r_k(A, B, C). \] (17)
Theorem 4.8 in [6]. Let $A \in \mathbb{R}^{n \times n}$ be Hurwitz and $B \in \mathbb{R}^{n \times l}$, $C \in \mathbb{R}^{q \times n}$. The following statements are equivalent:

(i) $\rho < r_\sigma(A, B, C)$,
(ii) there exists a CQLF for the set of matrices

$$\{A + BAC \mid \|A\|_2 \leq \rho\}.$$

To apply the above theorem it is necessary to define the radius $\rho$. The most important consequence of the decomposition made above (13) is that the perturbation matrix is actually a vector, and its spectral norm defines in fact a circle in $\mathbb{R}^2$. So the search for the stability radius may be approximated by mapping, taking increasing values for $\rho$ and checking if every static system along the circumference generated by each $\rho$ is Nyquist-stable. When the first unstable system appears, the radius is defined, like in Fig. 5. Then, as a consequence of the theorem, every arbitrary switching between parameter values inside the circle assures exponential stability.

Notice that the terms of the perturbation matrix are real. Consequently, the radius defined by mapping is $r_\sigma$. If $r_\sigma \neq r_\sigma$, here appears a source of conservativeness. In future research this question will be analyzed.

Note: Sometimes the designer finds the problem represented in Fig. 6. The zero to be varied is near the origin, and crossing the imaginary axis may represent instability. On the other hand, the variable pole is far away, and its range of variation is initially much higher.

The problem is that the stability result given before restricts the pole movement to a little circle defined by the limited variation range of the zero. That is, the pole and the zero are considered with the same scale. To overcome this, $L'(s)$ is represented in zero-pole-gain form, so the expression for the open loop transfer function becomes

$$L(s) = \frac{(s + z_1)}{(s + p_1)} L'(s) = \frac{k(s + z_2)(s + z_3)\ldots}{(s + p_2)(s + p_3)\ldots}.$$

(18)

Considering the $k$-gain of $L'(s)$ as part of the separated controller during the stability analysis,

$$\frac{k(s + z_1)}{(s + p_1)},$$

(19)

the perturbation matrix becomes $[\Delta P_1, k\Delta z_1]$. Then, as can be seen in Fig. 7, the stability radius actually defines an ellipse in the parameter space $(\Delta P_1, \Delta z_1)$. Consequently, the designer has much more freedom selecting the variation ranges for the pole and the zero.

IV. APPLICATION TO THE ACC BENCHMARK PROBLEM

Numerous solutions have been published for the two-mass-spring system (Fig. 8) ACC’92 benchmark problem [7]. Two of them have been selected here because their structure includes a real pole and a real zero, and also because the application of the proposed method provides a higher improvement in the transient response.

The linear controllers to be modified are

$$G_a(s) = \frac{2246.3(s + 0.237)(s^2 - 0.681s + 1.132)}{s(33.19)(s + 11.79)(s^2 + 4.95s + 7.563)},$$

(20)
obtained in [8], and
\[
G_b(s) = \frac{12.5 (s+0.203)(s-1.23)}{(s+20.6)(s^2 + 1.212s + 1.464)},
\]
selected from [9]. The poles and zeros to be varied and its range of variation are shown in Table I. It has been checked that the parameter variation is done inside the stability radius.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Original system</th>
<th>Fast system (high</th>
<th>Precise system (low</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_a)</td>
<td>(s = -0.237)</td>
<td>(s = -0.157)</td>
<td>(s = -0.317)</td>
</tr>
<tr>
<td>(p_a)</td>
<td>(s = -33.19)</td>
<td>(s = -37.19)</td>
<td>(s = -29.19)</td>
</tr>
<tr>
<td>(z_b)</td>
<td>(s = 0.203)</td>
<td>(s = -0.173)</td>
<td>(s = -0.233)</td>
</tr>
<tr>
<td>(p_b)</td>
<td>(s = -20.6)</td>
<td>(s = -24.6)</td>
<td>(s = -16.6)</td>
</tr>
</tbody>
</table>

The nonlinear switching functions employed appear in Fig. 9. In spite of their simplicity, the results show an important improvement, which anyway could be increased with more complex functions. In each case, it has been used the same function to move the pole and the zero \((N_1 = N_2)\).

![Switching functions](image1)

Figure 9. (a) Switching functions for controller \(G_a(s)\). (b) Switching functions for controller \(G_b(s)\).

It is simulated the response of the nominal system \((m_1 = m_2 = k = 1)\) to a unit impulse disturbance \((w_1, w_2)\) both in body 1 and 2 (problem 1 in [7]) and to a unit step command tracking (problem 4). Considerations about uncertainty in the problem are not taken into account, because it would be necessary a stability theory joining plant uncertainty (slow parameter variation) with controller switching (fast parameter variation). This may be a subject of future research.

A. First example results.

![Case a. Response to disturbance \(w_1\)](image2)

![Case a. Response to disturbance \(w_2\)](image3)

![Case a. Response to command tracking](image4)
B. Second example results.

![Figure 13](image1.png)

Figure 13. Case b. Response to disturbance $w_1$.

![Figure 14](image2.png)

Figure 14. Case b. Response to disturbance $w_2$.

![Figure 15](image3.png)

Figure 15. Case b. Response to command tracking.

The new strategy improves the results of the two previous controllers $[G_a(s) \text{ and } G_b(s)]$ for both the disturbance rejection and the reference tracking problems.

V. CONCLUSIONS

This paper presented a simple formulation to improve the performance of any linear controller. The methodology is independent of the technique used to design the previous controller. It consists of the segregation of the controller equation in two parts: a constant coefficients term and a variable expression that depends on the error amplitude. A practical stability condition for the variable controller is also introduced. The effectiveness of the method has been shown in a well-known benchmark problem, improving the results of two previous designs. Any control engineer, even not being a specialist on switching control, could easily use the method.

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