Assistance control based on a composite Lyapunov function for lane departure avoidance

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Abstract—This paper presents a vehicle steering assistance designed to avoid lane departure during driver inattention periods. Activated for a driver loss of concentration during a lane keeping maneuver the steering assistance drives the vehicle back to the center of the lane. In order to ensure a vehicle trajectory as close as possible to the centerline, the control law has been developed based on invariant sets theory and on composite Lyapunov functions. The computation has been performed using LMI methods, which allow in addition imposing a maximum bound for the control steering angle.

Index Terms—Steering angle control, level set, composite Lyapunov function, LMI

I. INTRODUCTION

Two innovative applications of the lateral control of the vehicle are the autonomous highway driving and the driver steering assistance [1], [2], [3].

In this paper we present a steering angle control law given by a composite Lyapunov function. The final goal is to design a steering assistance that maintains the vehicle on the lane during inattention periods of the driver. The assistance switches on if the driver is not alert and if, at the same time, he is heading towards a dangerous situation, defined as lane departure. A driving monitoring system could be used to identify the driver lack of attention. The assistance shall remain cooperative and shall not replace the driver. This means that, once activated, the automatic control shall switch off whenever the driver shows the intention to take control of the vehicle.

The present paper is a study of a feedback control law and focuses on the periods of time while the assistance is activated. For details on the switching strategy see [7]. One important concern of this type of assistance is how to control the trajectory followed by the vehicle during its activation periods. We talk here about trajectory in a large sense, in which it is included the fact that the vehicle shall stay near the center of the lane (during the activation of the assistance), having at the same time bounded and comfortable dynamics [7]. The control law proposed here ensures a vehicle trajectory that doesn’t exceed during the assistance activation some acceptable limits for the state variables.

In the next section we introduce the vehicle model. Our requirements concerning the assistance are synthesized in Section III. The definition of composite Lyapunov function and theorems necessary for the control design are given in Section IV. The design of the steering control law is presented in Section V. We perform in Section VI simulations of the controlled vehicle and we wrap up the paper in Section VII with the conclusions.

II. VEHICLE MODEL

As the study concerns the lateral control of a vehicle, the classical fourth order linear model (“bicycle model”) is used [4]. We assume in the present paper for the control design a straight road, road curvature \( \rho_{ref} = 0 \) (which is realistic for highways). The state vector is \( x = [\beta, r, \psi_L, \psi_L]^T \) and contains the following components: the vehicle side slip angle (\( \beta \)), the yaw rate (\( r \)), the relative yaw angle (\( \psi_L \)) and the lateral offset (\( \psi_L \)) from the centerline, taken at a lookahead distance \( l_S \). The state space model is the following:

\[
\begin{align*}
\dot{x} &= A \cdot x + B \cdot \delta_f + B_p \cdot \rho_{ref} \\
\dot{z} &= x
\end{align*}
\]

where \( \delta_f \) is the steering angle of the front wheels and

\[
A = \begin{pmatrix}
a_{11} & a_{12} & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 \\
0 & 1 & 0 & 0 \\
\nu & l_S & 0 & 0
\end{pmatrix},
B = \begin{pmatrix}
b_1 \\
b_2 \\
0 \\
0
\end{pmatrix},
B_p = \begin{pmatrix}
0 \\
0 \\
-\nu \\
0
\end{pmatrix}.
\]

The quantities in matrices \( A \) and \( B \) are given by

\[
\begin{align*}
a_{11} &= -2(\frac{c_f + c_r}{m}) \\
a_{12} &= -1 + 2(\frac{1}{m} \cdot \frac{c_f}{c_r}) \\
a_{21} &= 2\left(\frac{1}{m} \cdot \frac{c_f}{c_r}ight) \\
a_{22} &= -2\left(\frac{1}{m} \cdot \frac{c_f}{c_r}\right) \\
c_f &= c_{0f} \cdot \nu \\
c_r &= c_{0r} \cdot \nu \\
b_1 &= \frac{2c_r}{m} \\
b_2 &= \frac{2c_f}{m}
\end{align*}
\]

Table II in the Appendix shows nominal values for the above parameters.

Remark It can be easily shown that the system (1) is controllable except for a longitudinal speed \( \nu \) equal to zero. Having two poles at the origin the system is not stable.

III. REQUIREMENTS CONCERNING THE ASSISTANCE SYSTEM

We discuss in this section the qualitative control goals related to our assistance system.

We consider “normal driving” the following set of system states: \( |\beta| \leq \beta^N, |r| \leq r^N, |\psi_L| \leq \psi_L^N, |\psi_L| \leq \psi_L^N \), where \( \beta^N, r^N, \psi_L^N \) and \( \psi_L^N \) are some positive constant limits. This corresponds to a lane keeping situation with bounded vehicle dynamics. Thus, for a “normal driving” situation, the state vector \( x \) is supposed to be in the hypercube...
The main goal of the assistance developed in this paper is to avoid the lane departure in case the driver loses attention during “normal driving” \((x \in L(Z^N))\). We assume that the assistance control activates as soon as the driver loses attention, but only for a “normal driving” situation.

Therefore we require first a control that stabilizes asymptotically the system to a zero steady state, in particular to the centerline. Moreover, during its action the assistance shall provide a small overshoot with respect to the “normal driving” zone such that the vehicle trajectory does not exceed a “security zone”, for example avoid an overshoot that would lead the vehicles outside the road. In addition, the steering angle should remain bounded during the assistance intervention.

The “security zone” can be described similarly to the “normal driving” zone: \(|\beta| \leq \beta^M, |r| \leq r^M, |\psi_L| \leq \psi_L^M, |\psi_R| \leq \psi_R^M\), with \(\beta^M, r^M, \psi_L^M\) and \(\psi_R^M\) constant and positive. As specified above, we can associate to the “security zone” a hypercube \(L(Z^M)\) (see Section IV-A for definition).

IV. COMPOSITE QUADRATIC LYAPUNOV FUNCTION

In order to explain the design of the control law in Section V we introduce in this Section some definitions and theorems.

A. Definitions

We work in this paper with ellipsoidal and polyhedral sets and we explain their notations in the sequel.

A polyhedral set is defined using a matrix \(H \in \mathbb{R}^{m \times n}\), where we denote the rows of \(H\) by \(h_i\):

\[
L(H) := \{x \in \mathbb{R}^n : |h_i x| \leq 1, i = 1, \ldots, m\}. \tag{4}
\]

Thus the “normal driving” hypercube \(L(Z^N)\) is defined by:

\[
L(Z^N) := \{x \in \mathbb{R}^n : |z_i^N| x_i \leq 1, i = 1, \ldots, 4\}, \tag{5}
\]

where \(z_i^N \in \mathbb{R}^{1 \times 4}\) and \(z_i^N = (\beta^N) - 1, s_{1,2}^N = (r^N) - 1, z_i^M = (\psi_L^N) - 1, s_{1,2}^M = (\psi_R^N) - 1, z_i^j = 0\) for \(i = 1, \ldots, 4, i \neq j\).

Similarly the “security zone” is formalized as following:

\[
L(Z^M) := \{x \in \mathbb{R}^n : |z_i^M| x_i \leq 1, i = 1, \ldots, 4\}, \tag{6}
\]

where \(z_i^M \in \mathbb{R}^{1 \times 4}\) and \(z_i^M = (\beta^M) - 1, s_{1,2}^M = (r^M) - 1, z_i^M = (\psi_L^M) - 1, s_{1,2}^M = (\psi_R^M) - 1, z_i^j = 0\) for \(i = 1, \ldots, 4, i \neq j\).

For a matrix \(P \in \mathbb{R}^{n \times n}\) positive definite and symmetric an ellipsoidal set is denoted by:

\[
\varepsilon(P, \rho) := \{x \in \mathbb{R}^n : x^T Px \leq \rho\}, \quad \rho \geq 0. \tag{7}
\]

For \(\rho = 1\) the notation is simplified to \(\varepsilon(P) = \varepsilon(P)\).

Let us consider the positive definite and symmetric matrices \(P_1, P_2, \ldots, P_N \in \mathbb{R}^{n \times n}\). The set \(\Gamma\) contains vectors \(\gamma\) such that:

\[
\Gamma = \{\gamma \in \mathbb{R}^N : \sum_{j=1}^N \gamma_j = 1, \gamma_j \geq 0, j = 1, \ldots, N\}, \tag{8}
\]

where \(\gamma_j\) denotes the \(j\)-th element of \(\gamma\). For a vector \(\gamma \in \mathbb{R}^N\) the following matrices are defined:

\[
Q(\gamma) := \sum_{j=1}^N \gamma_j Q_j, \quad P(\gamma) := Q^{-1}(\gamma), \tag{9}
\]

where \(Q_j = P_j^{-1}, j = 1, \ldots, N\). Since the matrices \(Q_j\) are symmetric and positive definite for \(j = 1, \ldots, N\) the matrices \(Q(\gamma)\) and \(P(\gamma)\) are also symmetric and positive definite.

The composite quadratic function is defined by the authors of [8] as follows:

\[
V_c(\gamma) := \min_{\gamma \in \Gamma} x^T P(\gamma) x \tag{10}
\]

and is a positive definite function. Its level set is given by:

\[
\Gamma_{V_c}(\rho) := \{x \in \mathbb{R}^n : V_c(\gamma) \leq \rho\}. \tag{11}
\]

B. Theorems

The composite quadratic function defined in eq. (10) has as level set the convex hull of the \(x^T P x\) ellipsoids. Moreover it is continuously differentiable. These properties are formulated in the next theorem.

Theorem 4.1: [8]

1) \(
L(\gamma) = \rho \{x \in \mathbb{R}^n : V_c(\gamma) \leq \rho\}, \quad \Gamma_{V_c}(\rho) \tag{12}
\)

2) The function \(V_c(x)\) is continuously differentiable. Let \(\gamma'(x)\) be an optimal \(\gamma\) such that \(x^T P(\gamma'(x)) x = \min_{\gamma \in \Gamma} x^T P(\gamma) x\), then

\[
\frac{\partial V_c}{\partial x} = 2P(\gamma'(x)). \tag{13}
\]

Let us consider for an LTI system linear feedback vectors \(F_j\) associated with quadratic Lyapunov functions characterized by the matrices \(P_j\), for \(j = 1, \ldots, N\). We can compute a composite feedback law \(u = F x\) that stabilizes the LTI system and allows to the composite quadratic function from eq. (10) to be a Lyapunov function. The next theorem is adapted according to Theorem 4 given by [8], which establishes a saturated composite feedback law.

Theorem 4.2: [10] Let us consider the LTI system \(\dot{x} = Ax + Bu\) with \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times 1}\). Let be given ellipsoids \(\varepsilon(P_j)\) and feedback vectors \(F_j \in \mathbb{R}^{1 \times n}\), \(j = 1, \ldots, N\), such that

\[
(A + B F_j)^T P_j + P_j (A + B F_j) < 0, \quad \forall x \in \mathbb{R}^n \tag{14}
\]

and

\[
|F_j x| \leq D M, \quad \forall x \in \varepsilon(P_j), \tag{15}
\]

where \(DM\) is a positive constant. Let \(Q_j = F_j^{-1}, J = F_j Q_j\). For \(\gamma \in \Gamma\) define \(F(\gamma) := Y(\gamma) Q^{-1}(\gamma)\), where

\[
Y(\gamma) := \sum_{j=1}^N \gamma_j Q_j, \quad Q(\gamma) = \sum_{j=1}^N \gamma_j Q_j. \tag{16}
\]

Let \(\gamma'(x)\) be such that

\[
V_c(\gamma) = \min_{\gamma \in \Gamma} x^T P(\gamma) x = x^T P(\gamma'(x)) x. \tag{17}
\]
1) If the vector function $\gamma'(x)$ is continuous, then $u = F(\gamma'(x))x$ is a continuous feedback law.

2) $L_V(1) = \{x \in \mathbb{R}^n : V_c(x) \leq 1\}$ is contractively invariant under the feedback control law $u = F(\gamma'(x))x$.

3) In addition $|F(\gamma'(x))x| \leq DM \forall x \in L_V(1)$.

The continuity of the feedback law $u = F(\gamma'(x))x$ depends on the continuity of the vector function $\gamma'(x)$. Conditions for the continuity of $\gamma'(x)$ are stated in the sequel.

**Theorem 4.3:** [9] If none of the following is true then $\gamma'(x)$ is continuous

1) there exists $c \in \mathbb{R}^n$ and $j_1, j_2, \ldots, j_{n+1} \in \{1, \ldots, N\}$, $j_i \neq j_k$ for $i \neq k$, $i, k, 1, \ldots, n+1$ satisfying
   \[ c^T P_{j_1}^{-1}c = c^T P_{j_2}^{-1}c = \ldots = c^T P_{j_{n+1}}^{-1}c = 1. \]  
   (17)

2) there exists $c \in \mathbb{R}^n$, $j_1, j_2, \ldots, j_n \in \{1, \ldots, N\}$ and $\alpha_i$, $i \in \{1, \ldots, n\}$ satisfying
   \[ c^T P_{j_1}^{-1}c = c^T P_{j_2}^{-1}c = \ldots = c^T P_{j_n}^{-1}c = 1, n \leq n, \]
   \[ \sum_{i=1}^{n} \alpha_i P_{j_i}^{-1}c = 0, \quad \sum_{i=1}^{n} \alpha_i^2 = 1. \]  
   (18)

**V. DESIGN OF THE STEERING CONTROL LAW**

Summarizing the steering control requirements, we state that the assistance shall switch on for a driver lack of attention that occurs inside the “normal driving” zone $L(Z^N)$ and shall control the vehicle to the center of the lane keeping its trajectory inside the “security zone” $L(Z^{2M})$.

To achieve this result we have designed for system (1) a steering control that ensures invariant sets close to the “normal driving” zone and contained in the “security zone”. More precisely we have approximated the hypercube $L(Z^N)$ using the convex hull of eight ellipsoids $P_j, j = 1, \ldots, 8$, which are included in the hypercube $L(Z^M)$ and have as a main axe one of the eight diagonals of $L(Z^N)$ (see Fig. 1). For these ellipsoids we have computed feedback vectors $F_j$ such that the functions $x^T P_j x$ are Lyapunov functions for the closed loop system. We have further used the control law defined in Theorem 4.2 to ensure that the referred convex hull is consequently invariant, and then that all controlled trajectories that begin in $L(Z^N)$ won’t exceed the security zone $L(Z^{2M})$.

**A. Stage 1: Computation of the feedback vectors $F_j$ and of their associated Lyapunov functions $x^T P_j x$**

In a first step we have approximated the hypercube $L(Z^N)$ using the convex hull of eight ellipsoids $e(P_j)$, $j = 1, \ldots, 8$. On one hand an ellipsoid can be degenerated to a segment reducing to zero all its semi-axes excepting one. On the other hand we can see a hypercube as the convex hull of all its diagonals. So, the idea is to search ellipsoids oriented on the diagonals of $L(Z^N)$ minimizing the other semi-axes (Fig. 1).

These ellipsoids have to be level curves for Lyapunov functions for the closed loop system with linear feedbacks $F_j$. That is, we need to find $P_j$ and $F_j$ that satisfy:

\[ (A + BF_j)^T P_j + P_j (A + BF_j) < 0. \]  
(19)

For an exposed face $E$ of $L_V(1)$ $N_0$ is the number of intersections of $E$ and ellipsoids $\partial e(P_j), j = 1, \ldots, N_0$ (see [9] for details).

Therefore we are searching simultaneously appropriate matrices $P_j$ and feedback vectors $F_j, j = 1, \ldots, 8$. We impose in addition that these feedback vectors yield bounded steering angles for the control inputs.

The above conditions can be expressed as an LMI problem with a linear cost function for each matrix $Q_j = P_j^{-1}$ and feedback vector $F_j (Y_j = F_j Q_j), j = 1, \ldots, 8$:

\[ \min \\ \text{trace}(Q_j) \]
\[ Q_j > 0, \]
\[ Y^T B^T + Q_j A^T + A Q_j + B Y_j < 0, \]
\[ Q_j v_j = \lambda_j v_j, \]  
(20)

In the LMI optimization problem (20) we minimize the trace of $Q_j$ in order to reduce the amplitude of the non interesting semi-axes.

The second LMI condition ensures the closed loop stability for the feedback vectors $F_j$.

In the third LMI condition, after having defined a vector $v_j$ and a scalar $\lambda_j$, we search $Q_j$ to have $v_j$ as its eigenvector and $\lambda_j$ as its eigenvalue.

The fourth LMI condition confines the ellipsoid $e(P_j)$ inside the polyhedron $L(F_j) = \{ x \in \mathbb{R}^n : |F_j x| \leq DM \}$, and bounds in this way the feedback signal to $DM$ for a steering control activation inside $e(P_j)$.

Considering the hypercube $L(Z^N)$ we obtain all its $2^4$ vertices drawing for each state variable the positive or the negative corresponding normal bound: $\pm y_1^N, \pm y_2^N, \pm y_3^N$. Among these vertices we have chosen 2 vertices that are not symmetrical with respect to the origin to be the eigenvectors $v_j, j = 1, \ldots, 8$. The length of a semi-axis of the ellipsoid $e(P_j)$ is $\sqrt{\lambda_j} = ||v_j||_2$.

![Fig. 1. The ellipsoidal and hypercube zones represented for the variables $\psi_2$ and $\psi_L$.](image-url)

After computing all the feedback vectors $F_j$ and the symmetric positive definite matrices $P_j, j = 1, \ldots, 8$, given by the LMI problems (20) all the hypothesis of the Theorem 4.2 are satisfied for the LTI system defined in eq. (1). Using this theorem we can conclude that the set $L_V(1) = \{ x \in \mathbb{R}^n : V_c(x) \leq 1 \}$ is contractively invariant under the feedback $u = \delta_j(x) (Y(x)) x$, where

\[ V_c(x) = \min_{j \in \{1, \ldots, 8\}} x^T P_j (\gamma(x)) x, \]  
(21)

\[ F(\gamma(x)) = Y(\gamma(x)) Q^{-1}(\gamma(x)) = \left( \sum_{j=1}^{8} y_j^T Y_j \right) \left( \sum_{j=1}^{8} y_j^T Q_j \right)^{-1}. \]  
(22)

and $\gamma(x)$ is the solution of the minimization problem (21). Moreover the theorem guarantees that a continuous vector function $\gamma(x)$ yields a continuous feedback law and that the feedback steering angle for the composite
control $\delta_f(x) = F(\gamma^*)x$ is bounded to $DM$ for any control activation in $L_{V_{L}}(1)$.

**B. Stage 2: Computation of the projections on the axes of the convex hull**

We will now have a look at the trajectories of the controlled vehicle, trajectories that start inside the “normal driving” zone $L(Z^N)$. We know from Theorem 4.1 that the set $L_{V_{L}}(1)$ is the convex hull of the ellipsoids $\varepsilon(P_j), j = 1, \ldots, 8$. Since the ellipsoids $\varepsilon(P_j), j = 1, \ldots, 8$ include by design method all the vertices of $L(Z^N)$, the hypercube $L(Z^N)$ is also included in $L_{V_{L}}(1)$. Consequently all trajectories that begin in $L(Z^N)$ are contained in the contractively invariant set $L_{V_{L}}(1)$ and they will remain inside it while the steering control is activated. Therefore we need to find out the maximum values of the vehicle state $x$ inside $L_{V_{L}}(1)$.

The projection of a convex set on one dimension is also a convex set. It is straightforward to prove that the projections of $L_{V_{L}}(1)$ on each of the four axes $\beta, r, \psi_L$, and $y_L$ don’t exceed the projections of $\varepsilon(P_j), j = 1, \ldots, 8$ (see Property 1 by \[9\]). So, the maximum values of the vehicle state variables $x$ inside $L_{V_{L}}(1)$ are given by the maximum values of the projections of $\varepsilon(P_j), j = 1, \ldots, 8$ on each of the four axes.

For instance let be $\beta_{\text{max}}$ the maximum absolute value of the side slip angle$^3$. Then:

$$\beta_{\text{max}} := \max \{ \beta : \beta \in \text{Projection}_{\alpha[0.0,0.0]}(\varepsilon(P_j)), j = 1, \ldots, 8 \}, \quad (23)$$

where $\alpha \in \mathbb{R}$. In the same way we define the maximum projections of $L_{V_{L}}(1)$ on the axes $r, \psi_L, y_L$, respectively $r_{\text{max}}, \psi_{L_{\text{max}}}$ and $y_{L_{\text{max}}}$. There is a simple formula to compute the maximum projection of an ellipsoid on one axis:

$$\max \{ x_k : x_k \in \text{Projection}_{\alpha[0, \ldots, 1, \ldots, 0]}(\varepsilon(P_j)) \} = \sqrt{Q_{k,k}}, \quad (24)$$

where the “1” is on the $k$-th position of the vector $[0, \ldots, 1, \ldots, 0]$, $x_k$ is the $k$-th element of $x$ and $Q_{k,k}$ is an element from the diagonal of $Q$, $Q = P^{-1}$.

By computing the maximum values of the projections of $\varepsilon(P_j), j = 1, \ldots, 8$ on the four axes we have obtained the maximum values of the vehicle state $x$ in $L_{V_{L}}(1)$. This gives us the image of the worst case of the vehicle trajectory during the control activation.

As a last step we have to verify that the invariant set $L_{V_{L}}(1)$ is contained in the security zone $L(Z^M)$. This can be achieved by verifying that the point $[\beta_{\text{max}}, r_{\text{max}}, \psi_{L_{\text{max}}}, y_{L_{\text{max}}}]{^T}$ is included in $L(Z^M)$. If this is not the case then we can slightly modify the bounds of the “normal driving” hypercube $L(Z^N)$ and repeat the procedure (Stage 1) till the condition on $L(Z^M)$ is verified.

**C. Stage 3: Computation of the feedback law**

\[8\] has proposed a computational method to solve the minimization problem (21) and to obtain the optimal value $\gamma^*(x)$. The composite quadratic Lyapunov function $V_c(x)$ can be written as:

$$V_c(x) = \min \{ \sigma : \sigma \geq x^T P(\gamma)x, \gamma \in \Gamma \}. \quad (25)$$

Using the Schur complement, the above function can be expressed as the result of an optimization problem with linear matrix constraints:

$$V_c(x) = \min \sigma \quad \left( \begin{array}{c} x \quad \Sigma_{j=1}^{8} \gamma_j Q_j \end{array} \right) \geq 0, \quad (26)$$

The matrix variables are the parameters $\gamma_j$, for $j = 1, \ldots, 8$, and the positive minimization value $\sigma$.

In each time step we obtain after solving (26) the optimal values $\gamma^*$, and thus the corresponding feedback vector

$$F(\gamma^*) = \left( \sum_{j=1}^{8} \gamma_j Q_j \right)^{-1}. \quad (27)$$

With the above results it is straightforward to compute the feedback control law $\delta_f(x) = F(\gamma^*)x$. We notice that the above optimization is in fact an online computation, which is performed in each time step.

**D. Stage 4: Verification of the continuity of the feedback law**

To avoid actuator wear it would be better if the steering angle control law were continuous. But, as stated in Theorem 4.2, it is not a necessary condition for the system stability.

To verify the continuity of the control law we would have to verify the conditions (17) and (18) for $n = 4$ and $N = 8$. That means by a roughly approximation that we have to solve $C_4^8$ systems of polynomial equations for condition (17) and $C_8^4 + C_4^8 + C_8^4$ systems of polynomial equations for the condition (18). More precisely we have to verify only the existence of one real solution for each of the above systems of polynomial equations, which could simplify the computation but not the number of systems to solve.

There are a number of software packages that solve polynomial systems (see \[11\]). However this problem requires a hard computation effort and we have chosen not to perform the verification of the continuity of the feedback law.

**VI. SIMULATION RESULTS**

The first step before computing the steering feedback law is to define numerical bounds for the “normal driving” zone and for the “security zone”. For an experienced driver following the center of the lane the limits of the vehicle state $x$ are quite low. Using recorded data from an equipped vehicle and statistical results given by \[12\] we have decided to adopt in this example the following values: $\beta^N = 0.3^\circ (0.0052rad), \psi^N_1 = 3^\circ /s (0.0873rad/s), \psi^N_2 = 1^\circ (0.0175rad), y^c_L = 10cm$ (see Table I). These values define the vertices of the “normal driving” hypercube $L(Z^N)$ and therefore define the eigenvectors $v_j$
and the eigenvalues $\lambda_j$ for the matrices $Q_j$, $j = 1, \ldots, 8$. The values of the vehicle state that define the “security zone” $L(Z^M)$ have been chosen in consideration of the passengers comfort and to avoid the lane overshoot (see Table I). Also for comfort reasons the maximum steering angle for the feedback control has been fixed to $DM = 2\pi = 0.350rad$.

With the above values we have used the software for semidefinite programming exposed in [6] to solve the LMI problems with a linear cost function given in eq. (20). We have thus obtained the eight feedback vectors $F_j$ and the corresponding Lyapunov functions $x^T P_j x$ for $j = 1, \ldots, 8$.

The projections of the invariant set $L_{V_j}(1)$ on the four axes are given in Table I. These results show that for any assistance activation inside $L(Z^M)$ the vehicle trajectory will not exceed $17cm$ for the lateral offset $y_L$, $10.15\ rad/s$ (0.1773rad/s) for the yaw rate $r$, $2\ rad$ (0.0350rad) for the relative yaw angle $\psi_j$ and $1\ rad$ (0.0189rad) for the side slip angle $\delta_j$ during the activation of the steering control. The computed maximum projections of the invariant set $L_{V_j}(1)$ are inside the security zone $L(Z^M)$, so we can go over to the simulation stage.

To see the system behavior for an activation at the limit of the “normal driving” zone we have chosen as initial state a $L(Z^N)$ vertex $[0.0052, 0.0873, 0.0175, 0.1]^T$. The transitory dynamics of the vehicle state are given in Figs. 2 and 3 and the control steering angle in Fig. 4. We notice that the side slip angle $\beta$ has a peak value of 0.009rad (0.57°) bellow the maximum expected value of the $L_{V_j}(1)$ projection (0.018rad). The yaw rate $r$ doesn’t exceed $0.15rad/s$ (8.59°/s) compared to $0.1773rad/s$ (10°/s), the maximum projection value. The relative yaw angle $\psi_j$ stays bellow 0.021rad (1.2°), less than 0.035rad (2°), value computed as the maximum projection of $L_{V_j}(1)$ on the $\psi_j$ corresponding axis. Finally the lateral offset $y_L$ achieves a maximum value of $15.5cm$ compared to $16.6cm$, the maximum expected value. The control steering angle $\delta_j$ registered a maximum value of 0.032rad (1.83°), which is bellow the defined maximum value of 0.035rad (2°). The optimal solution for the $\gamma$ vector as the result of the optimization problem in each time step is given in Figs. 5 to 8.

The system behavior in closed loop under the influence of the road curvature. We have chosen a segment of the test track from Satory (France) with a road curvature below 0.01m rad$^{-1}$ (road radius higher than 100m) (see Fig. 9(a)). The state reference to be tracked has been set to zero. The resulting steering angle control is given in Fig. 9(b). For this test track it doesn’t exceed 1.6° (0.028rad). The lateral offset remains below 20cm (see Fig.11(b)). Comfortable values have been also registered for the maximum side slip angle (0.85°, 0.015rad, see Fig. 10(a)), for the maximum yaw rate (8.6°/s, 0.15rad/s, see Fig. 10(b)) and for the maximum relative yaw angle (1.71°, 0.03rad, see Fig. 11(a)). The above values show a good system robustness against perturbations.

### Table I

<table>
<thead>
<tr>
<th>$L(Z^N)$</th>
<th>$\beta$ (rad)</th>
<th>$r$ (rad/s)</th>
<th>$\psi_j$ (rad)</th>
<th>$y_L$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0052</td>
<td>0.0873</td>
<td>0.0175</td>
<td>0.1000</td>
</tr>
<tr>
<td>Projections $L_{V_j}(1)$</td>
<td>0.0189</td>
<td>0.1773</td>
<td>0.0350</td>
<td>0.1667</td>
</tr>
<tr>
<td>$L(Z^M)$</td>
<td>0.0262</td>
<td>0.2618</td>
<td>0.0873</td>
<td>0.4</td>
</tr>
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</table>

### VII. Conclusions

In this paper the design of a lateral driving assistance has been investigated. The main goal of the developed steering assistance is to bring the vehicle to the center of the lane for diminished driver attention. Secondly, the assistance shall provide a vehicle trajectory inside a “security zone” for an activation in the “normal driving” zone.

To begin with, the two-wheels vehicle model has been briefly presented followed in Section III by a description...
of the control objectives. The composite Lyapunov function and its main properties, the differentiability and the level set as a convex hull of the composing ellipsoids, have been further exposed. Following that, a control law based on the composite Lyapunov function has been introduced. This control ensures the asymptotic stability of a LTI system by means of bounded control values.

In the next sections this type of assistance has been used to control the steering angle of a vehicle satisfying the enounced requirements. For the vehicle system eight ellipsoids have been chosen to be oriented on the diagonals of the “normal driving” zone $L(Z^V)$. An LMI optimization problem with a linear cost function has been stated in order to compute each of the eight ellipsoids together with a stabilizing linear feedback. An additional LMI condition ensures also a bounded control for each ellipsoid and linear feedback.

Once the method of the control design has been established, we have considered a numerical example. The computed control law has been simulated using the “bicycle” vehicle model. Two cases have been considered: a first case which is an initial value regulation and a second case which is a closed loop control with a perturbation input, the road curvature. The results have confirmed the theoretical conclusions and have shown good robustness performances.

**TABLE II**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>$c_{fr}$</td>
<td>40000 N/rad</td>
</tr>
<tr>
<td>$c_{rl}$</td>
<td>35000 N/rad</td>
</tr>
<tr>
<td>$J$</td>
<td>2454 kg·m²</td>
</tr>
<tr>
<td>$l_f$</td>
<td>1.05 m</td>
</tr>
<tr>
<td>$l_r$</td>
<td>1.56 m</td>
</tr>
<tr>
<td>$l_c$</td>
<td>1.4 m</td>
</tr>
<tr>
<td>$k$</td>
<td>5 m</td>
</tr>
<tr>
<td>$m$</td>
<td>1500 kg</td>
</tr>
<tr>
<td>$v$</td>
<td>20 m/s</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.13 m</td>
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<tr>
<td>$\nu$</td>
<td>1</td>
</tr>
<tr>
<td>$L$</td>
<td>7.5 m</td>
</tr>
</tbody>
</table>

**REFERENCES**