Minimum Copper Loss Position Control of Linear Synchronous Motors with Current Limits

Nattapon Chayopitak and David G. Taylor
Georgia Institute of Technology
School of Electrical and Computer Engineering
Atlanta, Georgia 30332–0250 USA
Email: nattapon@gatech.edu, david.taylor@ece.gatech.edu

Abstract—An optimal control problem arising in manufacturing applications, namely the minimization of copper losses produced by point-to-point positioning, is formulated and solved numerically. Three types of synchronous motors and four types of current constraint sets are considered.

I. INTRODUCTION

Linear motors are particularly well suited to certain types of manufacturing automation applications, since they do not require rotary-to-linear transmissions to provide linear motion. For example, the ability to provide direct-drive linear motion is advantageous in assembly machines that employ a gantry-type configuration in which three Cartesian motion axes are operated to locate parts. Assembly processes require repetitive point-to-point positioning and hence continuous motive forces and continuous excitation currents, which in turn generate heat and result in temperature rise. Productivity is increased by reducing travel times, but this requires larger forces and hence higher average power dissipation. This paper provides some tools that may be used to help determine the trade-off between productivity and temperature rise.

Within this context, this paper is the first to explore the role of current constraint sets in detail. For the sake of clarity, only 3-phase synchronous motors are considered and these are limited to their idealized representation wherein spatial harmonics and magnetic saturation are neglected. All three types of such motors are included in this study: permanent magnet (PM), variable reluctance (VR) and hybrid. Magnitude limits may be imposed on the current vector as a whole or on the current vector components individually, and the phase windings can be interfaced to a power source using a 3-wire or 6-wire connection. Consequently, there are four possible current constraint sets, all of which are included in this study. In order to maximize productivity subject to a limit on acceptable average power dissipation, an optimal control problem [1] is formulated and solved numerically for each of the permutations described above.

This paper extends the authors’ recent work in the following sense: [8] considered only PM and VR synchronous motors with spherical current limit set and 3-wire connection; and [2] considered a periodic version of the problem using Fourier series, for PM, VR and hybrid synchronous motors with bandwidth-limited currents as opposed to magnitude-limited currents. Less closely related works are [9] and [5] for permanent-magnet brush-commutated dc motors and [6] for separately-excited brush-commutated dc motors.

II. PROBLEM FORMULATION

A. System Description

The position control problem considered here involves a frictionless single-axis motion system. The motive force is provided by a synchronous motor, constructed from linear magnetic materials with geometry that yields spatially sinusoidal magnetic characteristics. The dynamic model is

\[ M \ddot{x} = f(u_q, u_d) \]

where \( x \) denotes position, \( M \) is the moving mass, and the force function depends on the type of synchronous motor according to

\[
\begin{cases}
K_a u_q & \text{PM motor} \\
K_r u_q u_d & \text{VR motor} \\
K_a u_q + K_r u_q u_d & \text{hybrid motor}
\end{cases}
\]

where \( K_a \) is the alignment force constant, \( K_r \) is the reluctance force constant and \( u_{q/d} \) are the \( q/d \)-axis current components. The point-to-point positioning task corresponds to the boundary conditions

\[
\begin{bmatrix}
x(0) \\
x(T)
\end{bmatrix} =
\begin{bmatrix}
0 \\
X
\end{bmatrix}
\]

where \( T \) is the travel time and \( X \) is the travel distance.

B. Current Coordinate Transformation

The current variables appearing in (1), along with another current variable \( u_0 \) known as the 0-axis current component, are related to the physical currents \((\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\) flowing through the three phase windings by a coordinate transformation of the form

\[
\begin{bmatrix}
u_d \\
u_q \\
u_0
\end{bmatrix} = S(x) \begin{bmatrix}
\tilde{u}_1 \\
\tilde{u}_2 \\
\tilde{u}_3
\end{bmatrix}
\]

where

\[
S(x) = \sqrt{3} \begin{bmatrix}
\cos x_1 & \cos x_2 & \cos x_3 \\
-\sin x_1 & -\sin x_2 & -\sin x_3 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

where \( x_j = \frac{\pi}{p} x + (j - 1) \frac{2\pi}{3} \) for \( j = 1, 2, 3 \) and \( p \) is the pitch which characterizes the spatial periodicity of the linear magnetic structure. For VR motors \( p \) is the tooth pitch; otherwise, \( p \) is the magnet pitch. An advantage of
this change of variables, which is apparent in the notation of (1), is that it removes the dependence of the force function \( f \) on \( x \).

Temperature rise is determined in part by the average power dissipation in the phase windings, given by

\[
P = \frac{R}{T} \int_0^T \tilde{u}^T \tilde{u} \, dt
\]

where \( R \) is the resistance of the phase windings. Since \( S(x) \) is an orthonormal matrix, it follows that

\[
\tilde{u}^T \tilde{u} = (S^T(x)u)^T (S^T(x)u) = u^T u
\]

for all \( x \). Hence, in terms of the current variables used in (1), average power dissipation is proportional to the integral of \( u^T u \); this fact will be exploited in the formulation of the minimum copper loss control problem.

C. Current Limits

The physical currents \( \tilde{u} \) must necessarily be limited; an ultimate limit is imposed by the power source, but another more-restrictive limit is imposed by the controller to guarantee safe operation. There is more than one way to introduce the limit imposed by the controller. If the limit is imposed on the current vector as a whole, then a spherical limit set

\[
\mathcal{U}_s := \{ \tilde{u} : \tilde{u}^T \tilde{u} \leq U^2 \} = \{ u : u^T u \leq U^2 \}
\]

is obtained; when expressed in terms of \( u \), the boundaries of \( \mathcal{U}_s \) are independent of \( x \). If the limit is imposed on the current vector components individually, then a cubical limit set

\[
\mathcal{U}_c := \{ \tilde{u} : |\tilde{u}_j| \leq U, \; j = 1, 2, 3 \} = \{ u : |S_j(x)u| \leq U, \; j = 1, 2, 3 \}
\]

(10)

(11)

is obtained, where \( S_j(x) \) denotes the \( j \)th row of \( S(x) \); when expressed in terms of \( u \), the boundaries of \( \mathcal{U}_c \) depend on \( x \).

Fig. 1 illustrates the geometrical representation of \( \mathcal{U}_c \) with respect to \( u \), for \( x = 0 \), \( x = p/6 \) and \( x = p/3 \). The tilted cube rotates about the \( u_0 \)-axis by angles \( 0^\circ \), \( 30^\circ \) and \( 60^\circ \). Fig. 2 shows the cross sections in the \( dq \)-plane corresponding to different values of \( u_0 \). At \( u_0 = 0 \), the cross section is a regular hexagon; for \( |u_0| < U/\sqrt{3} \), it is a hexagon; for \( |u_0| \geq U/\sqrt{3} \), it is a triangle; and for maximum value \( |u_0| = \pm U/\sqrt{3} \), it is a point.

D. Wiring Connections

Three-phase synchronous motors have three phase windings and hence six wire ends for interfacing to a power source. From the perspective of power source design, a 6-wire connection is more versatile but a 3-wire connection is more economical. The 3-wire connection requires the free wire end from all phase windings to be interconnected as a floating node, so in this case the currents are confined to the set

\[
\mathcal{U}_0 := \{ \tilde{u} : \tilde{u}_1 + \tilde{u}_2 + \tilde{u}_3 = 0 \} = \{ u : u_0 = 0 \}
\]

(12)

(13)

Considering \( \mathcal{U}_c \), the \( dq \)-plane cross section is circular, and its radius is maximum at \( u_0 = 0 \); when \( u \in \mathcal{U}_c \) the choice \( u_0 = 0 \) is advantageous for producing large forces with small losses, and the use of a 3-wire connection would not be a restriction. Considering \( \mathcal{U}_s \), however, the \( dq \)-plane cross sections favorable for producing large forces generally correspond to \( u_0 \neq 0 \), e.g. the “largest” cross section generated by sweeping \( x \) corresponds to \( |u_0| = U/\sqrt{3} \); this type of operation requires a 6-wire connection, and also contributes an additional component to losses, so when \( u \in \mathcal{U}_s \), the optimal excitation is not easily determined.

E. Current Constraint Sets

In summary, with two types of current limits and two types of wiring connections, the four current constraint sets to be imposed on the optimal control problem will be generically denoted by \( \mathcal{U} \) and defined by

\[
\mathcal{U} = \left\{ \begin{array}{l}
\mathcal{U}_s \cap \mathcal{U}_0, \; \text{vector limit w/ 6-wire} \\
\mathcal{U}_c \cap \mathcal{U}_0, \; \text{vector limit w/ 3-wire} \\
\mathcal{U}_c \cap \mathcal{U}_0, \; \text{scalar limit w/ 3-wire} \\
\mathcal{U}_c \cap \mathcal{U}_0, \; \text{scalar limit w/ 6-wire}
\end{array} \right\}
\]

(14)

According to (9) and (11), \( \mathcal{U}_s \) and \( \mathcal{U}_c \) have been defined in terms of a common parameter \( U \) such that \( \mathcal{U}_s \subset \mathcal{U}_c \). Hence, \( \mathcal{U} = \mathcal{U}_c \) represents the least restrictive case.

III. OPTIMAL CONTROL PROBLEM

A. Minimum Time Control

The minimum possible travel time \( T^* \), along with the associated average power dissipation \( P^* \), is obtained by solving the optimal control problem

\[
\begin{align*}
\text{minimize} & \quad T = \int_0^T 1 \, dt \\
\text{subject to} & \quad (1), (3), \; u \in \mathcal{U}
\end{align*}
\]

(15)

(16)

Although the solution of this problem gives the minimal possible travel time \( T^* \), it also results in the maximum average power dissipation \( P^* \) [8]. If the allowable average power dissipation is restricted to \( P < P^* \), to limit temperature rise, then the problem formulation given below is more appropriate.

B. Minimum Copper Loss Control

Since \( T^* \) is the minimum achievable travel time for given values of \( M, K_a, K_r, U \) and \( X \), there will be infinitely many solutions to the positioning problem if \( P < P^* \). For this case, the objective here is to find the unique solution, from among all possible solutions, that minimizes \( T > T^* \) subject to an equality constraint on \( P \). Direct solution of this power-lossspecified time-optimal control problem would require an additional state equation to account for the equality constraint on \( P \). To avoid the additional complexity, it is possible to solve instead the optimal control problem

\[
\begin{align*}
\text{minimize} & \quad P = \frac{R}{T} \int_0^T u^T u \, dt \\
\text{subject to} & \quad (1), (3), \; u \in \mathcal{U}
\end{align*}
\]

(17)

(18)
Fig. 1. Cubical current limit set rotating at 0°, 30° and 60° about the \( u_0 \)-axis for \( U = 1 \, \text{A} \).

Fig. 2. Cubical current limit set in \( dq \)-plane at different values of \( u_0 \) for \( U = 1 \, \text{A} \).
for fixed $T > T_*$. This problem formulation yields the same family of solutions as the power-loss-specified time-optimal control when $T$ is varied as a parameter \[8\].

**IV. NUMERICAL METHOD**

Consider the general optimal control problem

\[
\begin{align*}
& \text{minimize} \quad J = \int_0^T L(x, u) \, dt \quad (19) \\
& \text{subject to} \quad \dot{x} = F(x, u) \quad (20) \\
& \quad x(0) = x_0, \; x(T) = x_T \quad (21) \\
& \quad u \in U \quad (22)
\end{align*}
\]

which covers (17)–(18) as a special case. Approximate solutions may be computed by formulating a nonlinear programming problem using collocation [3], [4], [7]. Trajectories $u(t)$ and $x(t)$ are approximated on a mesh $0 = t_1 < t_2 < \cdots < t_N = T$ with spacing $h_j = t_{j+1} - t_j$, such that $u_j \approx u(t_j)$ and $\hat{x}_j \approx x(t_j)$. The nonlinear programming problem will iterates the vector

\[
y = \begin{bmatrix} \hat{u}_1^T & \cdots & \hat{u}_N^T & \hat{x}_1^T & \cdots & \hat{x}_N^T \end{bmatrix}^T
\]

and $u(t)$ and $x(t)$ will be recovered by interpolation. For $u(t)$ the interpolation formula

\[
u(t) \approx \hat{u}_j + \frac{t - t_j}{h_j} (\hat{u}_{j+1} - \hat{u}_j), \quad t_j \leq t \leq t_{j+1}
\]

guarantees that $u(t)$ interpolates to computed values $\hat{u}_j$ at $t = t_j$. For $x(t)$ the interpolation formula is

\[
x(t) \approx \sum_{i=0}^3 c_i \left( \frac{t - t_j}{h_j} \right)^i, \quad t_j \leq t \leq t_{j+1}
\]

To guarantee that $x(t)$ interpolates to computed values $\hat{x}_j$ at $t = t_j$, it is necessary to impose the constraints

\[
c_{0,j} = \hat{x}_j
\]

\[
c_{0,j} + c_{1,j} + c_{2,j} + c_{3,j} = \hat{x}_{j+1}
\]

To guarantee that $x(t)$ satisfies the differential equation at $t = t_j$, it is necessary to impose the constraints

\[
c_{1,j}/h_j = \hat{F}_j
\]

\[
(c_{1,j} + 2c_{2,j} + 3c_{3,j})/h_j = \hat{F}_{j+1}
\]

where $\hat{F}_j = F(\hat{x}_j, \hat{u}_j)$. Solving the combined system of constraint equations leads to

\[
c_{0,j} = \hat{x}_j
\]

\[
c_{1,j} = h_j \hat{F}_j
\]

\[
c_{2,j} = 3(\hat{x}_{j+1} - \hat{x}_j) - h_j (\hat{F}_j + 2\hat{F}_{j+1})
\]

\[
c_{3,j} = -2(\hat{x}_{j+1} - \hat{x}_j) - h_j (\hat{F}_j + \hat{F}_{j+1})
\]

To formulate differential equation constraints at mesh midpoints, note that at $t_{c,j} \approx \frac{1}{2}(t_j + t_{j+1})$

\[
u(t_{c,j}) \approx \frac{1}{2} (\hat{u}_j + \hat{u}_{j+1}) =: \hat{u}_{j+\frac{1}{2}}
\]

\[x(t_{c,j}) \approx c_{0,j} + \frac{1}{2} c_{1,j} + \frac{1}{4} c_{2,j} + \frac{1}{8} c_{3,j} =: \hat{x}_{j+\frac{1}{2}}
\]

\[\dot{x}(t_{c,j}) \approx \left( c_{1,j} + c_{2,j} + \frac{1}{2} c_{3,j} \right)/h_j =: \hat{x}_{j+\frac{1}{2}}
\]

**V. NUMERICAL RESULTS**

The parameter values are chosen as follows: $M = 1$ kg, $X = 0.05$ m, $p = 0.01$ m, $R = 5 \Omega$, $U = 5$ A, $K_s = 4$ N/A for the PM motor, $K_r = 1.6$ N/A$^2$ for the VR motor and $(K_{sa}, K_{sr}) = (1.238$ N/A, $1.238$ N/A$^2$) for the hybrid motor. The force constants are chosen such that each motor has the same maximum force of 20 N for $u \in U_s$. Numerical solutions of (17)–(18) for $T > T_*$ are obtained for four cases of $U$. The numerical method uses $N = 41$; due to discretization, it is unable to reproduce the jump discontinuities in current known to characterize the analytical solution for the VR motor with $u \in U_s$.

Plots of position, velocity, force and currents for each synchronous motor, with $u \in U_s$ and either 3-wire or 6-wire connection, are shown in Fig. 3. Both 3-wire and 6-wire connection yield identical results. All three motors accelerate and decelerate with constant ripple-free forces, but otherwise the shape of the force trajectory depends on the type of motor. All the motors operate with $u_0 = 0$ which is consistent with previous discussions. Values of average power dissipation $P < P_s$ for various values of $T > T_*$. are listed in Table I. In this case, for any fixed $T$, the VR motor generates the most heat and the PM motor generates the least heat.

Plots of position, velocity, force and currents for each synchronous motor, with $u \in U_s$, are shown in Fig. 4 for 3-wire connection and in Fig. 5 for 6-wire connection. The 3-wire and 6-wire results differ from each other, with higher peak forces for the 6-wire connection. In both cases all three motors accelerate and decelerate with nonconstant forces exhibiting ripple, and the shape of the force trajectory depends on the type of motor. Values of

<table>
<thead>
<tr>
<th>$T$ [s]</th>
<th>PM</th>
<th>Hybrid</th>
<th>VR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125</td>
<td>38.4473</td>
<td>48.9770</td>
<td>50.4405</td>
</tr>
<tr>
<td>0.120</td>
<td>45.2691</td>
<td>54.9892</td>
<td>56.3573</td>
</tr>
<tr>
<td>0.115</td>
<td>53.7847</td>
<td>62.5023</td>
<td>63.7484</td>
</tr>
<tr>
<td>0.110</td>
<td>64.9417</td>
<td>72.3381</td>
<td>73.4401</td>
</tr>
<tr>
<td>0.105</td>
<td>81.0885</td>
<td>86.6996</td>
<td>87.4481</td>
</tr>
<tr>
<td>0.100*</td>
<td>125</td>
<td>125</td>
<td>125</td>
</tr>
</tbody>
</table>

* Analytical results from [8].
average power dissipation $P < P^*$ for various values of $T > T^*$, are listed in Tables II and III for the 3-wire and 6-wire connections. For any fixed $T$, $u \in U_c$ yields lower cost than $u \in U_s$, and $u \in U_c$ with 6-wire connection yields lower cost than $u \in U_c$ with 3-wire connection. These results are consistent with previous discussions. For
any fixed $T$, with $u \in U_c$ the VR motor generates the least heat and the PM motor generates the most heat.

Solutions to the time-optimal control problem (15)–(16) are approximated by solving (17)–(18) as follows. First, a sufficiently large value of $T$ is chosen so that the nonlinear programming problem will converge. Then successively smaller values of $T$ are chosen and the nonlinear programming problem is run to check for convergence. This procedure repeats until convergence fails to occur, and $T^*$ is approximated by the previous value of $T$ in the sequence. The results obtained using this method are listed in Table IV. The tabulated results using $u \in U_c$ are reasonably accurate in comparison with analytical results reported in [8]; approximation errors in $P_*$ are less than 5%. As expected, $T^*$ is smallest for $u \in U_c$.

**VI. Conclusion**

The minimum copper loss position control problem has been formulated and solved numerically for a class of synchronous motors. Study of this problem is motivated by manufacturing applications in which the performance of electric motors in continuous point-to-point positioning tasks is often limited by power dissipation. As anticipated from the construction of constraint sets, when the travel distance and travel time are both specified, the numerical results show that use of the cubical current limit with 6-wire connection yields the lowest power dissipation for all three types of synchronous motors. When only travel distance is specified, the time-optimal control solutions have been obtained in an indirect way using the proposed problem formulation.

**REFERENCES**


