Robust Discrete-Time Simple Adaptive Tracking

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Abstract—This paper provides, rigorous results on robust output-feedback simplified adaptive controllers tracking for discrete-time systems with uncertainties. Sufficient conditions for the closed-loop stability and perfect tracking of the proposed simplified adaptive control scheme, given in the paper, lead to almost strictly positive realness requirement on the plant under certain conditions. The numerical example demonstrates the proposed method.

I. INTRODUCTION

Adaptive control methods cope with unknown/changing plant parameters by adjusting the control law to the varying plant on-line. They may be divided into explicit (indirect) control, which separately applies plant-parameters identification and control schemes, and implicit (direct) control, where the control gains are directly computed (without identifying the plant parameters). Simplified Adaptive Control (SAC) is a class of direct adaptive controller schemes which has received considerable attention in the literature for continuous-time systems ([1],[2],[3]). Robustness of SAC controllers facing polytope uncertainties has already been established ([4],[5],[6]) and continuous-time SAC has reached a reasonable degree of maturity (see especially reference [3]), allowing application to real engineering problems (see e.g. reference [6]). The stability of continuous-time SAC is related to the Strictly Positive Real (SPR) property of the controlled plant. There are some recent results and statements regarding discrete-time set-ups ([7]-[11]), but in general discrete-time output-feedback SAC seems yet far from the degree of maturity of continuous-time SAC. Fu and Cheng ([9]) have suggested a unique gain adaptation formula and have established the stability of the resulting state-feedback discrete-time SAC in the presence of bounded real uncertainties. Bar-Kana ([11]) has recently provided a proof of the fact that any proper minimum-phase discrete-time linear system with positive definite input-output feed-through matrix D is Almost Strictly Positive Real (ASPR).

In the present paper, a rigorous Lyapunov-based framework for the tracking and stabilization of discrete-time uncertain systems by SAC output-feedback controllers is developed, and sufficient conditions are derived for the stability of the closed-loop and the gain adaptation formula. These sufficient conditions are expressed in terms of Bilinear Matrix Inequalities (BMI) which can be solved using local iterations. When the ASPR requirement is replaced by the more restrictive SPR requirement, Linear Matrix Inequalities (LMIs) are obtained. Numerical example are given which illustrate these method.

Throughout the paper the superscript ‘T’ stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space, $\mathbb{R}^{n \times m}$ is the set of all n x m real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n \times n}$ means that P is symmetric and positive definite. The trace of a matrix $Z$ is denoted by $\text{tr} \{ Z \}$, $\text{diag} \{ a, b \}$ denotes a diagonal matrix with a and b on the diagonal and $G^*(s) \equiv G^T(-s)$. The convex hull defined by the polytope vertices $\Omega_j$, $j = 1, ...N$ is denoted by $\text{co} \{ \Omega_1, ..., \Omega_j \}$ and $\text{col} \{ a, b \}$ for vectors a and b denotes the augmented vector $[a^T, b^T]^T$. In symmetric block matrices we use * as an ellipsis for terms that are induced by symmetry.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following discrete-time linear system:

\[
\begin{align*}
    x_p(k+1) &= A_p x_p(k) + B_p u_p(k) \\
    y_p(k) &= C_p x_p(k) + D_p u_p(k)
\end{align*}
\]

where $x_p(k) \in \mathbb{R}^n$ is the system state, $y_p(k) \in \mathbb{R}^m$ is the plant output, $u_p(k) \in \mathbb{R}^m$ is the control input and $A_p$, $B_p$, $C_p$ and $D_p > 0$ are constant matrices of appropriate dimensions.

Remark 1: In the general case, for any proper but not strictly proper system, when $D_p$ is not positive definite but satisfies $D_p + D_p^T > 0$, we define $u_p(k) = D_p^T \tilde{u}_p(k)$ and obtain the following representation for (1)

\[
\begin{align*}
    x_p(k+1) &= A_p x_p(k) + \tilde{B}_p \tilde{u}_p(k) \\
    y_p(k) &= C_p x_p(k) + \tilde{D}_p \tilde{u}_p(k)
\end{align*}
\]

where $\tilde{B}_p = B_p D_p^T$ and $\tilde{D}_p = D_p D_p^T > 0$. We therefore, assume in the sequel without loss of generality that $D_p > 0$.

The plant (1) is required to follow the output of the asymptotically stable model:

\[
\begin{align*}
    x_m(k+1) &= A_m x_m(k) + B_m u_m(k) \\
    y_m(k) &= C_m x_m(k) + D_m u_m(k)
\end{align*}
\]

where $x_m(k) \in \mathbb{R}^q$ is the system state, $y_m(k) \in \mathbb{R}^m$ is the plant output, $u_m(k) \in \mathbb{R}^m$ is the control input and $A_m$, $B_m$, $C_m$ and $D_m$ are constant matrices of appropriate dimensions. The reference model (3) is used to define the desired input-output behavior of the plant and then important to note that the dimension of the reference model state may be less than the dimension of the plant state. However, since $y_p(k)$ is to track $y_m(k)$, the number of model outputs is must be equal to number of plant output. Asymptotically Perfect Tracking (APT) is defined as tracking with zero steady-state tracking error

\[y_p(k) = y_m(k)\]
Lemma 2: Asymptotically Perfect Tracking (APT) is possible if the system of (1) is ASPR.

Proof: Equation (1b) can be written in the form of:

\[ u_p(k) = D_p^{-1}(y_m(k) - C_p x_p(k)) \] (4)

Substituting (4) in (1a), gets:

\[ x_p(k + 1) = (A_p - B_p D_p^{-1} C_p) x_p(k) + B_p D_p^{-1} y_m \]

Therefore, we need \( A_p - B_p D_p^{-1} C_p \) to be stable, but it is guaranteed by the fact that \( (A_p, B_p, C_p, D_p) \) is ASPR.

Suppose that the system of (1) is ASPR and then APT is possible.

The next lemma determines that a relation exists between the plant's state and the model's state and input with any additional signal.

Lemma 3: For any auxiliary arbitrary input signal \( \tilde{u}(k) \), there exist \( F(k) \in \mathbb{R}^{n \times q} \), \( G(k) \in \mathbb{R}^{n \times m} \) such that the trajectories of (1) are of the form:

\[ x_p(k) = F(k) x_m(k) + G(k) u_m(k) + \tilde{u}(k) \] (5)

Proof: Suppose that the \( x_m(0) = 0 \) and then we get:

\[ x_m(k) = \sum_{i=0}^{k-1} A_m^{k-1} B_m u_m(i) \] (6)

and similarly we get:

\[ x_p(k) = A_p^{k-1} x_p(0) + \sum_{i=0}^{k-1} A_p^{k-i} B_p u_p(i) \] (7)

Next, substituting (7) and (6) in (5) gets:

\[ Y(k) = X(k) U(k) + \tilde{u}(k) \nabla^{T}(k) \]

where:

\[ Y(k) = A_p^{k-1} x_p(0) + \sum_{i=0}^{k-1} A_p^{k-i} B_p u_p(i) \]

\[ X(k) = [ F(k) \ G(k) \] \nabla^{T}(k) = \left[ \left( \sum_{i=0}^{k-1} A_p^{k-i} B_m u_m(i) \right)^{T} \left( u_m(k) \right)^{T} \right] \]

and finally:

\[ X(k) = (Y(k) - \tilde{u}(k)) U(k) \]

As (8) has \( n \) equations with \( n \times (q + m) \) variables and then the existence of the \( F(k) \) and \( G(k) \) is satisfied in general.

Define:

\[ e_y(k) = y_m(k) - y_p(k) \]

namely,

\[ e_y(k) = C_m x_m(k) + D_m u_m(k) - C_p x_p(k) - D_p u_p(k) \] (9)

Next, substituting (5) in (9) gets:

\[ e_y(k) = (C_m - C_p F(k)) x_m(k) + (D_m - C_p G(k)) u_m(k) - D_p u_p(k) - C_p \tilde{u}(k) \]

or finally, obtain:

\[ u_p(k) = K_p^* e_y(k) + K_p^* x_m(k) + K_u^* y_m(k) - \tilde{u}(k) \] (10)

where \( K_p^* \equiv -D_p^{-1} K_p \), \( K_u^* \equiv D_p^{-1}(C_m - C_p F(k)) \), \( K_u^* \equiv D_p^{-1}(D_m - C_p G(k)) \) and \( \tilde{u}(k) \equiv D_p^{-1} C_p \tilde{u}(k) \).

If the ideal control \( u_p^*(k) \) defined as:

\[ u_p^*(k) = K_p^* x_m(k) + K_u^* y_m(k) \] (11)

feeds the system:

\[ x_p^*(k + 1) = A_p x_p^*(k) + B_p u_p^*(k) \]
\[ y_p^*(k) = C_p x_p^*(k) + D_p u_p^*(k) \] (12a,b)

then by substituting (11) in (12b) we obtain:

\[ y_p^*(k) = y_m \]

Therefore, the ideal control \( u_p^*(k) \) allows APT, namely \( e_y \equiv 0 \). For the more general case where the latter is not satisfied, we seek a controller of the form:

\[ u_p(k) = K^* r(k) - \tilde{u}(k) \] (13)

where:

\[ K^*(k) = [ K_p^* \ K_u^* \ K_u^* ] \] (14)

\[ r(k)^T = [ e_y(k) \ x_m(k) \ u_m(k) ] \] (15)

where \( e_y(k) = y_m(k) - y_p(k) \) is an error between the model output and the output of the plant, \( K_p^* \in \mathbb{R}^{m} \), \( K_u^* \in \mathbb{R}^{m \times q} \) and \( K_u^* \in \mathbb{R}^{m} \) are stabilizing and bounded gains (due to stability of (3) and for bounded \( u_m \) and \( \tilde{u}(k) \) is an auxiliary input signal. This control, however, require calculation of the \( F(k) \) and \( G(k) \) and requires the explicit knowledge of the system dynamics.

Instead, we use the direct adaptive control scheme known as the Simplified Adaptive Control(SAC) [3] to calculate the gains which lead, in the steady state, to the same control signal that would be achieved by \( K_p^* \), \( K_u^* \) and \( K_u^* \). It is noted here that the following adaptive control law requires neither the explicit knowledge of the gains matrix \( K_p^* \), \( K_u^* \) and \( K_u^* \), nor the knowledge of the system dynamics.

III. Calculating of the Stabilizing Gains

Consider the following direct adaptive control scheme known as the Simplified Adaptive Control(SAC) [3]:

\[ u_p(k) = K(k) r(k) \] (16)

where:

\[ K(k) = [ K_e(k) \ K_u(k) \ K_u(k) ] \] (17)

and where

\[ K(k) = K(k - 1) + e_y(k)r^T(k) \] (18)

The initial condition is \( K(0) = 0 \). We Define \( \delta(k) = K^*(k) - K(k) \), that is the difference between the ideal control \( K^*(k) \) and the current SAC gain \( K(k) \). The control law of (16) can be now expressed by the following choice of the auxiliary control signal of (13):

\[ \tilde{u}(k) = \delta(k)r(k) \] (19)
Defining $\eta(k) = K(k) - K(k-1) = e_u(k) \tau^T(k)$, which is the gain increment between steps, and $\Delta K^*(k) = K^*(k) - K^*(k-1)$, which is the ideal gain increment, we obtain that:

$$\delta(k) - \delta(k-1) = \Delta K^*(k) - \eta(k). \quad (20)$$

and thus the following holds.

$$\delta(k)e^T(k) = \delta(k)\tau(k)e^T_y(k) = \bar{u}(k)e^T_y(k) \quad (21)$$

Next, following [3], we define the state and output errors:

$$e_x(k) = x^*_p(k) - x_p(k)$$
$$e_y(k) = y^*_p(k) - y_p(k)$$

Defining also the control error $e_u(k) = u^*_p(k) - u_p(k)$ and using (13) and (11), we obtain:

$$e_u(k) = -K^*_e e_y(k) + \bar{u}(k) \quad (22)$$

which, after simple algebraic manipulations, leads to:

$$e_x(k+1) = A_pe_x(k) + B_pe_u(k)$$
$$e_y(k) = C_pe_x(k) + D_pe_u(k) \quad (23a, b)$$

Substituting (23b) in (22) we obtain:

$$e_u(k) = -K^*_e (C_pe_x(k) + D_pe_u(k)) + \bar{u}(k) \quad (24)$$

In order to resolve the algebraic loop in (24), we define

$$\tilde{K}_e^* = (I + \tilde{K}_e^* D_p)^{-1} K^*_e$$

and note that $(I + \tilde{K}_e^* D_p)^{-1} = (I - K^*_e D_p)$. The algebraic loop in (24) thus results in:

$$e_u(k) = -\tilde{K}_e^* C_pe_x(k) + (I - \tilde{K}_e^* D_p)\bar{u}(k). \quad (25)$$

Substituting in (23) we obtain:

$$e_x(k+1) = (A_p - B_p \tilde{K}_e^* C_p)e_x(k) + B_p(I - \tilde{K}_e^* D_p)\bar{u}(k)$$
$$e_y(k) = (I - D_p \tilde{K}_e^* D_p)e_x(k) + D_p(I - \tilde{K}_e^* D_p)\bar{u}(k)$$

We define $\tilde{A}_p = A_p - B_p \tilde{K}_e^* C_p$, $\tilde{C}_p = (I - D_p \tilde{K}_e^* D_p)$, $\tilde{B}_p = B_p(I - \tilde{K}_e^* D_p)$ and $\tilde{D}_p = D_p(I - \tilde{K}_e^* D_p)$ and obtain:

$$e_x(k+1) = \tilde{A}_pe_x(k) + \tilde{B}_p\bar{u}(k)$$
$$e_y(k) = \tilde{C}_pe_x(k) + \tilde{D}_p\bar{u}(k) \quad (26a, b)$$

We are now in position to state the main result of this section:

**Theorem 4:** For an ASPR plant, the adaptive scheme consisting of the plant (1), the control law (16) and the gain adaptation formula (18), create bounded gains and states for any input command, and APT is attained if the reference signal constant.

**Proof:** In order to establish the desired APT of (1), the global asymptotic stability (GAS) of the closed-loop system of (26) should be proven. We consider the following radially-unbounded Lyapunov function candidate

$$V(e_x(k), K(k)) = e^T_x(k)P e_x(k) + tr\{\delta(k-1)\delta^T(k-1)\} + \delta^T(k)\delta(k)$$

Note that $V(0, K^*(k)) = 0$, $V(e_x(k), K(k)) > 0$ for all $(e_x(k), K(k)) \neq (0, K^*(k))$ and $V(e_x(k), K(k)) \rightarrow \infty$ as $\|e_x(k)\| \rightarrow \infty$ or $K(k) \rightarrow \infty$. The Lyapunov difference along the closed-loop system trajectories is given by:

$$\Delta V_k = V_{k+1} - V_k = e_x(k+1)^T Pe_x(k+1) - e_x(k)^T Pe_x(k) + tr\{\delta(k)\delta^T(k)\} - tr\{\delta(k-1)\delta^T(k-1)\}$$

where we have suppressed the dependence of $V_k$ on $e_x(k)$ and $K(k)$ for the sake of simplicity. Substituting (20), we have:

$$\Delta V_k = e_x(k+1)^T Pe_x(k+1) - e_x(k)^T Pe_x(k) + tr\{\delta(k)\delta^T(k)\} - tr\{\delta(k)\delta^T(k)\}$$

namely,

$$\Delta V_k = e_x(k+1)^T Pe_x(k+1) - e_x(k)^T Pe_x(k) - 2tr\{\delta(k)\eta(k)\} + 2tr\{\delta(k)\Delta K^*(k)\}$$

Substituting (29) in (29), gets:

$$\Delta V_k = e_x(k+1)^T Pe_x(k+1) - e_x(k)^T Pe_x(k) - 2tr\{\delta(k)\eta(k)\} + 2tr\{\delta(k)\Delta K^*(k)\}$$

Using (23) and (21) and the fact that $tr(AB) = tr(BA)$, we have:

$$\Delta V_k = (\tilde{A}_p e_x(k) + \tilde{B}_p\bar{u}(k))^T P(\tilde{A}_p e_x(k) + \tilde{B}_p\bar{u}(k)) - e_x(k)^T Pe_x(k) - (\tilde{C}_p e_x(k) + \tilde{D}_p\bar{u}(k))^T \bar{u}(k)$$

namely,

$$\Delta V_k = \lambda_1(k) + 2tr\{\delta(k)\} + 2tr\{\delta(k)\Delta K^*(k)\} + 2tr\{\delta(k)\Delta K^*(k)\}$$

where:

$$\lambda_1(k) = [ e^T_x(k) \bar{u}(k) ] \Gamma_1 [ e_x(k) \bar{u}(k) ] + e^T_y(k)\Gamma_2 e_y(k)$$

$$\Gamma_1 \Delta = \begin{bmatrix} \tilde{A}_p^T & -P & -\tilde{A}_p^T & \tilde{A}_p^T \\ \bar{P} & \bar{P} & -\bar{P} & \bar{P} \\ -D_p & \bar{D}_p & \bar{D}_p & \bar{D}_p \\ \tilde{B}_p^T & \tilde{B}_p^T & \tilde{B}_p^T & \tilde{B}_p^T \end{bmatrix}$$

$$\Gamma_2 \Delta = -r(k)$$

**Remark 5:** If the system of (26) is SPR, $\Gamma_1 < 0$ and, therefore $\lambda_1(k) \leq 0$. Since:

$$\Delta K^*(k) = [ 0 \quad \Delta K^*_e(k) \quad \Delta K^*_p(k) ]$$

(33)
substituting $\delta(k) = K^*(k) - K(k)$ and using (33) and (15), we find that:

$$\Delta V_k = \lambda_1(k) + 2tr(e_k^T(k)x_m(k)\Delta K^*(k)^T + e_k^T(k)u_m(k)\Delta K^*(k)^T - tr(\Delta K^*(k)\Delta K^*(k)^T) + 2tr((K^*_u - K_u)\Delta K^*_u(k)^T + (K^*_u - K_u)u_m(k))\Delta K^*_u(k)^T)$$

(34)

Note that $K_k(k) = \sum_{i=0}^{m} e_i(i)x_m(i)$ and $K_u(k) = \sum_{i=0}^{m} e_i(i)u_m(i)$ and then rewrite the (34) as:

$$\Delta V_k = \lambda_1(k) + \lambda_2(k)$$

(35)

where:

$$\lambda_2(k) = 2tr(e_k^T(k)x_m(k)\Delta K^*_u(k)^T + e_k^T(k)u_m(k)\Delta K^*_u(k)^T - tr(\Delta K^*(k)\Delta K^*(k)^T) + 2tr((K^*_u - K_u)\Delta K^*_u(k)^T + (K^*_u - K_u)u_m(k))\Delta K^*_u(k)^T)$$

(36)

We next show that the system states error and gains are all bounded. If system (1) is ASPR, $\lambda_1(k) \leq 0$. However, $\Delta V(k)$ may not be negative semidefinite since $\lambda_2(k)$ is non-definite. The latter, $\lambda_2(k)$ is (only) an affine function of $e_k(k)$, whereas $\lambda_1(k)$ is quadratic in $e_k(k)$. Hence, if $e_k(k)$ becomes large the term $\lambda_1(k)$ becomes dominant. Then $\Delta V(k)$ becomes negative, guaranteeing ($\Delta K^*$ is bounded) that all adaptation variables are bounded (LaSalle’s Invariance Principle, see e.g. [3]).

At steady-state the input command $u_m(k)$ and the states $x_m(k)$ and $x_p(k)$ do not change. Thus, at steady-state $\Delta K^*(k) = 0$ and $\lambda_2(k) = 0$, so we have:

$$\Delta V_k = \lambda_1(k) \leq 0$$

(37)

The equality $\Delta V_k = 0$ is achieved only when $e_x(k) = 0$, $e_y(k) = 0$ and $\tilde{u}(k) = 0$. Namely, $V_k(k) \geq 0$, where $V_k(k)$ is radially unbounded and monotonically decreases whenever $e_x(k) \neq 0$ or $e_y(k) \neq 0$ or $\tilde{u}(k) \neq 0$. It therefore follows that from any $v_k(0) \in R^a$ and $K(0) \in R^{3n+m}$ the state-vectors $e_x(k)$ and $K(k)$ will tend, as $k \to \infty$, to $\xi_k$ and $K^*(k)$, thus satisfying $\Delta V(k) = 0$ which by (37) can only happen when $e_x(k) = 0$ and $e_y(k) = 0$ and $\tilde{u}(k) = 0$. Thus, APT is attained if the reference signal becomes constant.

Remark 6: From the definitions of $\delta(k)$ and $r(k)$ it follows that

$$\tilde{u}(k) = \delta(k)r(k) = (K^*_u - K_u(k))e_y(k) + (K^*_u - K_u(k))x_m(k) + (K^*_u - K_u(k))u_m(k)$$

When APT is attained, $e_y(k) = 0$ and $\tilde{u}(k) = 0$. Then,

$$e_y(k) + (K^*_u - K_u(k))x_m(k) + (K^*_u - K_u(k))u_m(k) = 0$$

(38)

Therefore, APT does not necessarily imply $K^*_u(k) = K(k)$.

Remark 7: The radially unboundedness condition assure [12] that the contour surfaces of $V(e_x(k), K(k))$ will correspond to closed curves. Namely, the state can not drift away from the equilibrium point, by going through contours corresponding to smaller and smaller $V^*$.

IV. ROBUST SIMPLIFIED ADAPTIVE CONTROL TRACKING

We next extend the result of Theorem 1 to the case where the $A$ matrix of the discrete-time system (1) are not exactly known. Denoting

$$\Omega = \{ A_p, B_p \}$$

(39)

where $\Omega \in Co\{\Omega_i, i = 1, ..., N\}$, namely,

$$\Omega = \sum_{i=1}^{N} f_i \Omega_i$$

(40)

where the vertices of the polytope are described by

$$\Omega_i = \{ A_p^{(i)}, B_p^{(i)} \}, \quad i = 1, 2, ..., N.$$ (41)

Theorem 8: If the plant (1) is ASPR at the vertices $\Omega_i$, then throughout $Co\{\Omega_i\}$ the adaptive scheme consisting of the plant (1), the control law (16) and the gain adaptation formula (18) leads to bounded gains and states for any input command, and APT is attained if the reference signal becomes constant.

Proof: The system (1) is bounded in the vertices if there exist $P > 0, K^*_u$ that satisfy the following set of BMIs:

$$\begin{bmatrix}
\tilde{A}_p^{(i)^T} & P \tilde{A}_p^{(i)} - P \\
\tilde{C}_p^{(i)^T} - \tilde{A}_p^{(i)^T} P \tilde{B}_p^{(i)} - 2\tilde{D}_p^{(i)^T} + \tilde{B}_p^{(i)^T} P \tilde{B}_p^{(i)}
\end{bmatrix} < 0,
\quad i = 1, 2, ..., N.$$ (42)

It is easy to show that we can rewrite the inequalities of (42) as:

$$\begin{bmatrix}
-P & \tilde{C}_p^{(i)^T} \\
* & -2\tilde{D}_p^{(i)^T} - \tilde{B}_p^{(i)^T} P - P
\end{bmatrix} < 0,
\quad i = 1, 2, ..., N.$$ (43)

where the latter is affine in $A_p^{(i)}$ and $B_p^{(i)}$. We thus readily obtain by multiplying (43) by $f_i$ and summing over $i = 1, 2, ..., N$ that $\Gamma_1 < 0$ is satisfied throughout $\Omega$.

Remark 9: If vertex dependent $K^*_u$ is allowed to reduce the conservatism in (43), then the result of Theorem 7 still holds if $\tilde{B}_p$ is restricted to be vertex independent.

Remark 10: It follows from the equivalence between the ASPR property and the minimum- phase (MP) property that if the plant is MP at the all of the vertices then the above theorem (6) holds. Namely, it can be establishes equivalence between the ASPR property of (1) which can be
verified by the BMI (42) and its minimum-phase property which can be verified by the following LMI:

$$\left( H(i)^T \right)^T P H(i) - P < 0 \ , \ i = 1, 2, ..., N. \quad (44)$$

where

$$H(i) = (A_p(i) - B_p D_p^{-1} C_p) \quad (45)$$

V. NUMERICAL EXAMPLES

In this section we present numerical example to demonstrate the application of the theory developed above.

A. Pitch Loop SAC APT of a missile short period dynamics

Consider a modified version of the angle of attack/pitch-rate dynamics example of [13]. This example describes the short period dynamics of a missile and was used in [13] to study gain scheduled control. A servo model with time constant of $1/\tau = 30 [rad/sec]$ was applied. The state-vector is $x = [\theta, \alpha, q, \delta]^T$ where $\theta, \alpha$ are the pitch angle and pitch rate respectively, $q$ [rad/sec] is the pitch rate angle and $\delta$ [rad] is the elevator angle. The plant input is the elevator angle command $\delta_{com}[rad]$, and the plant output is the pitch-angle plus 0.1 of pitch-rate plus 0.01 of $\delta_{com}$ where the latter terms were added in order to respectively improve the effective damping of the missile short period mode and to ensure the ASPR property of the open-loop discrete-time system. It should be also noted that the nonzero but small $D = 0.01$ has no particular physical significance. The plant is described by continuous time state-space model for $N = 4$, where:

$$A = \begin{bmatrix} -0.001 & 0 & 1 & 0 \\ 0 & -Z_{\alpha_j} & 1 & 0 \\ 0 & 0 & -M_{\alpha_j} & 0 \\ 0 & 0 & 0 & -1/\tau \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\tau \end{bmatrix} \quad (46)$$

and where the parameters of the four vertices (Mach-Altitude Pairs of $(0,5), (0.5, 18km), (4,0), (4, 18km)$ are $Z_{\alpha} \in \{0.5, 0.5, 4, 4\}$ and $M_{\alpha} \in \{6, 106, 6, 106\}$. As can be seen the uncertainty appears only in A. The first element on the diagonal of A is taken to be nonzero in order to allow a solution to the strict inequalities in (44). We assume that the pitch angle and pitch rate can be measured. Define:

$$C = \begin{bmatrix} 1 & 0 & 0.1 & 0 \end{bmatrix} \quad \text{and} \quad D = 0.01 \quad (47)$$

The states of this plant embedded within a zero-order-hold scheme (i.e. a sample and hold device on its input and a sampler on its output) using a discrete-time SAC with a sampling time of $T_s$ seconds. We, therefore, first derive the discrete-time version of the above continuous-time plant, assuming a zero-order hold at the plant input and taking a sampling-time of $T_s = 1/64$. Using (44) and Linear Matrix Inequalities we get the plant is M.P. in all vertices where the single

$$P = \begin{bmatrix} 4.3272 & -1.8886 & 0.1430 & 0.0039 \\ -1.8886 & 4.3103 & 0.0168 & -0.0022 \\ 0.1430 & 0.0168 & 0.0362 & 0.0010 \\ 0.0039 & -0.0022 & 0.0010 & 0.0001 \end{bmatrix} \quad (48)$$

was used to verify $(A - BD^{-1}C)^T P (A - BD^{-1}C) - P < 0$ at the above four vertices (or equivalently verifying the quadratic stability of the zero dynamics of the plant) we obtain that the plant is ASPR and system (46) with the control law (16) and the gain adaptation formula (18) is APT or PT for $T_s = 1/64 [Hz]$. The reference system is:

$$A = \begin{bmatrix} -3 & -10 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [0 \ 10] \quad (49)$$

Our aim is the plant output (pitch angle) track to reference model outputs for step command input to (49). Figure 1 depicts the input and output of the reference model.

The simulation results are given in figures 2a,b and 3a,b.

The initial conditions given $\theta = \alpha = 0^\circ$ and $q = 0^\circ/sec$. Fig. 2 describes the states $\theta, q$, and Fig. 3 describes the states $\alpha$ and the elevator angle command $(\delta_{com})$ for each time step of the all 4 operation points. Apparently, pitch angle which is the output of the plant model tracks the output of the reference model and all the other states are regulated to zero by the proposed the control law (16) and the gain adaptation formula (18). Somewhat a more sluggish tracking is observed for the second operating point which corresponds to the lowest dynamic pressure requiring the largest $D_p u$ which causes the largest $|\theta - y|$. 

VI. CONCLUSIONS

In this paper, the existing theory of Simplified Adaptive Control has been generalized for tracking problem of output feedback discrete-time systems. The results confirm the conjecture relating the closed-loop stability of APT systems to the almost strictly positive realness of the system or equivalently to its minimum phase property. A similar property, which can be verified using Linear Matrix Inequalities is shown to be valid also for system with polytopic uncertainties.

The results are illustrated via an example, from the field of flight control. The results encourage further research to analyze the effects of exogenous disturbance and measurement noise in simplified adaptive control schemes.

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Fig. 1. The input and output of the reference model

Fig. 2. Simulation Results at the 4 operating points (states 1,2)

Fig. 3. Simulation Results at the 4 operating points (states 3,4)